6

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2.4.4 Octahedral Stress

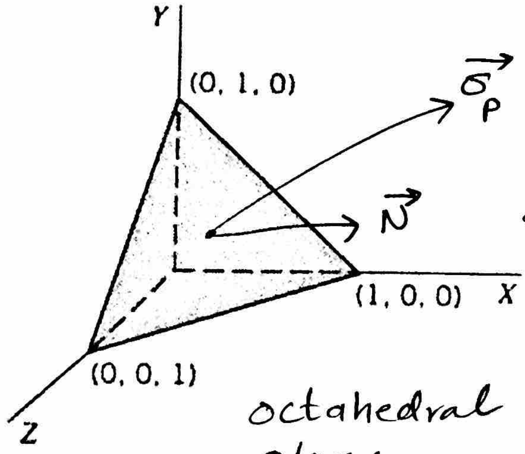
$$\vec{N} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$ (normalized $\vec{n} = \hat{i} + \hat{j} + \hat{k}$, $\vec{N} = \frac{\vec{n}}{\sqrt{3}}$)

$$\vec{\sigma_p} = \sigma_{px}\hat{i} + \sigma_{py}\hat{j} + \sigma_{pz}\hat{k}$$

$$\sigma_{px} = l\,\sigma_{xx} + m\,\sigma_{xy} + n\,\sigma_{xz}$$

$$\sigma_{py} = l\,\sigma_{xy} + m\,\sigma_{yy} + n\,\sigma_{zy}$$

$$\sigma_{pz} = l\,\sigma_{xz} + m\,\sigma_{yz} + n\,\sigma_{zz}$$



principal coordinate system (x, y, z)

(For principal coordinate system, $\sigma_{xx} = \sigma_1$, $\sigma_{yy} = \sigma_2$, $\sigma_{zz} = \sigma_3$ and all shear stress $= 0$)

Normal stress

$$\sigma_{pN} = \vec{\sigma_p} \cdot \vec{N} = \sigma_{oct} = \frac{1}{3} I_1 = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$= \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Shear stress

$$\sigma_{ps} = \sqrt{\sigma_p^2 - \sigma_{pN}^2} = \tau_{oct}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{xx} - \sigma_{zz})^2 + (\sigma_{yy} - \sigma_{zz})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)}$$

$$= \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2}$$

$$= \sqrt{\frac{2}{9} I_1^2 - \frac{2}{3} I_2}$$

2.4.5 Mean and Deviatric stress

For a stress state $[\sigma]$

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

The mean stress (σ_M) is

$$\sigma_M = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} I_1$$

then the mean stress matrix $[\sigma_m]$

$$[\sigma_m] = \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix}$$

Define the deviatric stress matrix $[\sigma_d]$

$$[\sigma_d] = [\sigma] - [\sigma_m]$$

$$[\sigma_d] = \begin{bmatrix} \sigma_{xx} - \sigma_M & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_M & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_M \end{bmatrix}$$

mean and deviatric stress are important in failure theories "Chapter 4"

For $[\sigma_d] = \begin{bmatrix} \sigma_{xx}-\sigma_M & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_M & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_M \end{bmatrix}$

If we want to find the principal values and principal directions of $[\sigma_d]$, we follow the previous procedure.

as

$$\sigma_d^3 - J_1 \sigma_d^2 + J_2 \sigma_d - J_3 = 0 \rightarrow \sigma_{d1} \geq \sigma_{d2} \geq \sigma_{d3}$$

principal values of $[\sigma_d]$

J_1, J_2 and J_3 are deviatric stress invariants

$$J_1 = tr(\sigma_d) = 0$$

$$J_2 = \frac{1}{2}\left(tr^2(\sigma_d) - tr(\sigma_d^2)\right), \quad [\sigma_d^2] = [\sigma_d][\sigma_d]$$

$$= -\frac{1}{6}\Big[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)\Big]$$

$$= -\frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right]$$

$$= I_2 - \frac{1}{3} I_1^2$$

$$J_3 = \det(\sigma_d) = (\sigma_1-\sigma_M)(\sigma_2-\sigma_M)(\sigma_3-\sigma_M)$$

$$= I_3 - \frac{1}{3} I_1 I_2 + \frac{2}{27} I_1^3$$

thus, the principal values of $[\sigma_d]$, are

① $\sigma_{d_1} = \sigma_1 - \sigma_M = \dfrac{2\sigma_1 - \sigma_2 - \sigma_3}{3} = \dfrac{(\sigma_1 - \sigma_3) + (\sigma_1 - \sigma_2)}{3}$

② $\sigma_{d_2} = \sigma_2 - \sigma_M = \dfrac{(\sigma_2 - \sigma_3) + (\sigma_2 - \sigma_1)}{3} = \dfrac{(\sigma_2 - \sigma_3) - (\sigma_1 - \sigma_2)}{3}$

③ $\sigma_{d_3} = \sigma_3 - \sigma_M = \dfrac{(\sigma_3 - \sigma_1) + (\sigma_3 - \sigma_2)}{3} = \dfrac{(\sigma_1 - \sigma_3) + (\sigma_2 - \sigma_3)}{3}$

" $\sigma_{d_i} = \sigma_i - \sigma_M$ "

From Eq (1), (2) and (3)

$$\Rightarrow \sigma_{d_1} + \sigma_{d_2} + \sigma_{d_3} = 0$$

* For principal directions

$\{V_{d_i}\} = \{V_i\}$, normalized $\{V_{dn_i}\} = \{V_{n_i}\}$

The principal directions of $[\sigma_d]$ are the same as those of $[\sigma]$.

Example: $[\sigma] = \begin{bmatrix} 30 & 25 & -10 \\ 25 & 10 & 20 \\ -10 & 20 & 5 \end{bmatrix}$

① Find $[\sigma_m]$

② Find $[\sigma_d]$

③ Find principal values are principal directions of $[\sigma_d]$
↳ Normalized

Solution

① $[\sigma_m] = \begin{bmatrix} \sigma_M & 0 & 0 \\ 0 & \sigma_M & 0 \\ 0 & 0 & \sigma_M \end{bmatrix}$, $\sigma_M = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$

$= \frac{1}{3}(30 + 10 + 5) = \frac{45}{3} = 15$

$[\sigma_m] = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

② $[\sigma_d] = [\sigma] - [\sigma_m'] = \begin{bmatrix} 15 & 25 & -10 \\ 25 & -5 & 20 \\ -10 & 20 & -10 \end{bmatrix}$

③ Principal values and directions

$\Longrightarrow$ It's easier to find them for $[\sigma]$

$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$

$I_1 = 45$, $I_2 = 1767$, $I_3 = -24625$

$\Rightarrow \sigma_1 = 47$, $\sigma_2 = 22$, $\sigma_3 = -24$

So,

$$\sigma_{d_1} = 47 - 15 = 32$$

$$\sigma_{d_2} = 22 - 15 = 7$$

$$\sigma_{d_3} = -24 - 15 = -39$$

- Then find principal directions for $[\sigma]$ as we learned in the last class and normalize

- The principal planes of $[\sigma_d]$ are the same of $[\sigma]$, as:

$$\{V_{dn_1}\} = \begin{Bmatrix} -0.090 \\ -0.600 \\ -0.804 \end{Bmatrix}$$

$$\{V_{dn_2}\} = \begin{Bmatrix} 0.788 \\ 0.453 \\ -0.417 \end{Bmatrix}$$

$$\{V_{dn_3}\} = \begin{Bmatrix} -0.609 \\ 0.671 \\ -0.424 \end{Bmatrix}$$