* Hyperbolic Equation (wave equation).

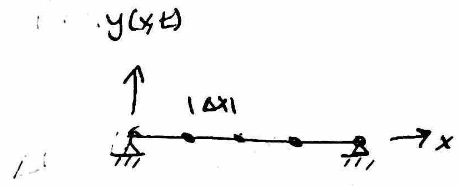
$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial x^2}$$

t: time, x: space, c: constant.

$t > 0$ $\quad 0 < x < L$ ← length

Let $b^2 = 1/c^2$ $\quad y(x,t)$.

$$\frac{\partial^2 y}{\partial t^2} - b^2 \frac{\partial^2 y}{\partial x^2} = 0$$



Divide x

Divide t $\quad \Delta t$

↳ 0 → t

Using centered Finite differences.

$$\frac{\partial^2 y}{\partial t^2} = \frac{y(x_i, t_{j+1}) - 2y(x_i, t_j) + y(x_i, t_{j-1})}{(\Delta t)^2}$$, Δt: time step size

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{y(x_{i+1}, t_j) - 2y(x_i, t_j) + y(x_{i-1}, t_j)}{(\Delta x)^2}$$, Δx: space step size

Substitute in the PDE

$$\frac{y_{i,j+1} - 2y_{i,j} + y_{i,j-1}}{(\Delta t)^2} - b^2 \frac{y_{i+1,j} - 2y_{i,j} + y_{i-1,j}}{(\Delta x)^2} = 0$$

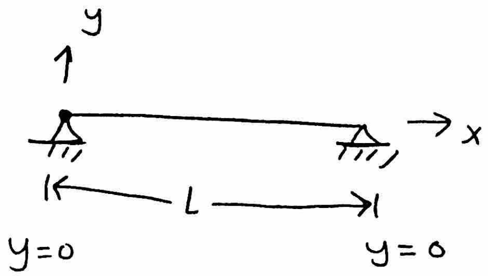
Define $\lambda = \frac{b\Delta t}{\Delta x}$, thus:

wave difference equation

$$y_{i,j+1} - 2y_{i,j} + y_{i,j-1} - \lambda^2 y_{i+1,j} + 2\lambda^2 y_{i,j} - \lambda^2 y_{i-1,j} = 0$$

$i, j = 0, 1, 2, 3, \ldots, n$

To solve wave equation we need 2 Boundary conditions and 2 initial conditions, as follows: ②



BC's

$y(0,t)=0$, $y(L,t)=0$

Initial conditions

$y(x,0)=f(x)$, $\frac{dy}{dt}(x,0)=g(x)$

For the $\frac{dy}{dt}(x,0)=g(x)$, use forward finite difference

$$\frac{dy}{dt}=\frac{y_{i,1}-y_{i,0}}{(\Delta t)} \Rightarrow y_{i,1}=y_{i,0}+\Delta t\,\frac{dy}{dt}$$

at $t=0 \Rightarrow dy/dt=g(x)$

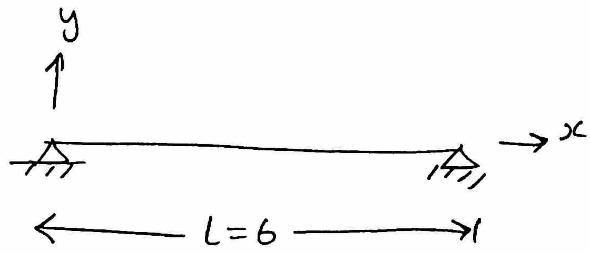
$$\Rightarrow y_{i,1}=y_{i,0}+\Delta t\, g(x_i) \quad , \quad y_{i,0}=y(x_i,0)=f(x_i)$$

$$y_{i,1}=f(x_i)+\Delta t\, g(x_i)$$

Example

For a simply supported string with length of 6 m shown,
If the governing equation is :

$$\frac{\partial^2 y}{\partial t^2} - b^2 \frac{\partial^2 y}{\partial x^2} = 0$$



Solve this system for $t = 0 \rightarrow 4$ seconds.
Use $\Delta t = 2$ seconds , $\Delta x = 2m$, $b = 2$

BC's $y(0,t) = 0$
$y(l,t) = 0$

IC's $y(x,0) = 0$
$y_t = \partial y/\partial t \, (x,0) = 5$

Solution

i	x_i
0	$x_0 = 0$
1	$x_1 = 2$
2	$x_2 = 4$
3	$x_3 = 6$

j	t_j
0	$t_0 = 0$
1	$t_1 = 2$
2	$t_2 = 4$

we know ① $\overset{=0}{x_0}$ at all t , From BC's , is always zero
② $\overset{=L=6}{x_3}$ at all t , From BC's , is always zero
③ x_1 and x_2 at $t_0 = 0$, From Initial condition, is always zero

$\Rightarrow$ we need x_1 at only $t_1 = 2$, $t_2 = 4$ } 4 unknowns
x_2 at only $t_1 = 2$, $t_2 = 4$ } 4 Equation

x_1, t_1 { $i=1, j=1$
x_1, t_2 { $i=1, j=2$

x_2, t_1 $i=2, j=1$
x_2, t_2 $i=2, j=2$

we know wave difference equation

$$y_{i,j+1} - 2y_{i,j} + y_{i,j-1} - \lambda^2 y_{i+1,j} + 2\lambda^2 y_{i,j} - \lambda^2 y_{i-1,j} = 0$$

or

$$y_{i,j+1} + 2(\lambda^2 - 1)y_{i,j} + y_{i,j-1} - \lambda^2 y_{i+1,j} - \lambda^2 y_{i-1,j} = 0$$

For point $i=1,\ j=1\ (x_1, t_1)$

$$y_{1,2} + 2(\lambda^2-1)y_{1,1} + y_{1,0} - \lambda^2 y_{2,1} - \lambda^2 y_{0,1} = 0$$

$$\lambda = \frac{b\Delta t}{\Delta x} = \frac{(2)(2)}{(2)} \Rightarrow \lambda = 2,\quad \boxed{\lambda^2 = 4}$$

we know $y_{0,1} = y(x_0, t_1) = 0$ (From BC's)

$y_{1,0} = y(x_1, 0) = 0$ (From IC's)

$$y_{1,1} = y_{1,0} + (\Delta t)\, g(x_1) = 0 + (2)(5) \Rightarrow y_{1,1} = 10$$

$$y_{2,1} = y_{2,0} + (\Delta t)(g(x_2)) = 0 + (2)(5) \Rightarrow y_{2,1} = 10$$

remember: $y_{i,1} = y_{i,0} + (\Delta t)\, g(x_i)$

$$\Rightarrow y_{1,2} + 2(4-1)10 - (4)(10) = 0$$

$$\Rightarrow y_{1,2} = -20$$

continue for all points, to solve for $y_{i,j}$.

Practice problem

Solve the following wave equation

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0 \quad , \quad 0 \le x \le 1 \qquad 0 \le t \le 6$$

with BC's

$$y(0,t) = u(1,t) = 0$$

IC's

$$y(x,0) = \sin(\pi x)$$

$$y_t(x,0) = 0$$

, Use $\Delta x = 0.2$, $\Delta t = 1$

i	x_i
0	$x_0 = 0$
1	$x_1 = 0.2$
2	$x_2 = 0.4$
3	$x_3 = 0.6$
4	$x_4 = 0.8$
5	$x_5 = 1$

j	t_j
0	$t_0 = 0$
1	$t_1 = 1$
2	$t_2 = 2$
3	$t_3 = 3$
4	$t_4 = 5$
5	$t_5 = 6$

For $y_{0,j} = 0$ BC's

$y_{5,j} = 0$ BC's

b. Use the temperature distribution of part (a) to calculate the strain I by approximating the integral

$$I = \int_{0.5}^{1} \alpha T(r, t) r \, dr,$$

where $\alpha = 10.7$ and $t = 10$. Use the Composite Trapezoidal method with $n = 5$.

12.3 Hyperbolic Partial Differential Equations

In this section, we consider the numerical solution to the **wave equation**, an example of a *hyperbolic* partial differential equation. The wave equation is given by the differential equation

$$\frac{\partial^2 u}{\partial t^2}(x, t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x, t) = 0, \quad 0 < x < l, \quad t > 0, \tag{12.16}$$

subject to the conditions

$$u(0, t) = u(l, t) = 0, \quad \text{for} \quad t > 0,$$

$$u(x, 0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x), \quad \text{for} \quad 0 \le x \le l,$$

where α is a constant dependent on the physical conditions of the problem.

Select an integer $m > 0$ to define the x-axis grid points using $h = l/m$. In addition, select a time-step size $k > 0$. The mesh points (x_i, t_j) are defined by

$$x_i = ih \quad \text{and} \quad t_j = jk,$$

for each $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots$.

At any interior mesh point (x_i, t_j), the wave equation becomes

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_j) = 0. \tag{12.17}$$

The difference method is obtained using the centered-difference quotient for the second partial derivatives given by

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} - \frac{k^2}{12} \frac{\partial^4 u}{\partial t^4}(x_i, \mu_j),$$

where $\mu_j \in (t_{j-1}, t_{j+1})$, and

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} - \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j),$$

where $\xi_i \in (x_{i-1}, x_{i+1})$. Substituting these into Eq. (12.17) gives

$$\frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} - \alpha^2 \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}$$
$$= \frac{1}{12}\left[k^2 \frac{\partial^4 u}{\partial t^4}(x_i, \mu_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\right].$$

Neglecting the error term

$$\tau_{i,j} = \frac{1}{12}\left[k^2 \frac{\partial^4 u}{\partial t^4}(x_i, \mu_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\right], \tag{12.18}$$

leads to the difference equation

$$\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{k^2} - \alpha^2 \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} = 0.$$

Define $\lambda = \alpha k/h$. Then we can write the difference equation as

$$w_{i,j+1} - 2w_{i,j} + w_{i,j-1} - \lambda^2 w_{i+1,j} + 2\lambda^2 w_{i,j} - \lambda^2 w_{i-1,j} = 0$$

and solve for $w_{i,j+1}$, the most advanced time-step approximation, to obtain

$$w_{i,j+1} = 2(1-\lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}. \tag{12.19}$$

This equation holds for each $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots$. The boundary conditions give

$$w_{0,j} = w_{m,j} = 0, \quad \text{for each } j = 1, 2, 3, \ldots, \tag{12.20}$$

and the initial condition implies that

$$w_{i,0} = f(x_i), \quad \text{for each } i = 1, 2, \ldots, m-1. \tag{12.21}$$

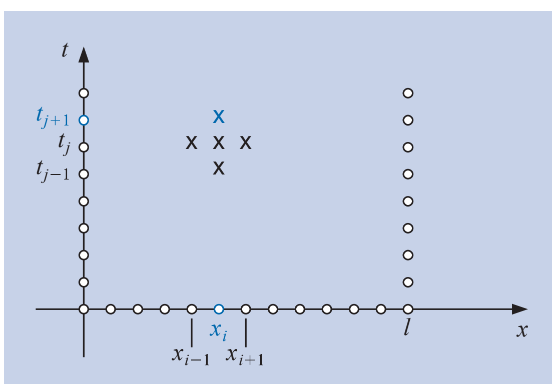
Writing this set of equations in matrix form gives

$$\begin{bmatrix} w_{1,j+1} \\ w_{2,j+1} \\ \vdots \\ w_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{m-1,j} \end{bmatrix} - \begin{bmatrix} w_{1,j-1} \\ w_{2,j-1} \\ \vdots \\ w_{m-1,j-1} \end{bmatrix}. \tag{12.22}$$

Equations (12.18) and (12.19) imply that the $(j+1)$st time step requires values from the jth and $(j-1)$st time steps. (See Figure 12.12.) This produces a minor starting problem because values for $j = 0$ are given by Eq. (12.20), but values for $j = 1$, which are needed in Eq. (12.18) to compute $w_{i,2}$, must be obtained from the initial-velocity condition

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \quad 0 \le x \le l.$$

Figure 12.12



One approach is to replace $\partial u/\partial t$ by a forward-difference approximation,

$$\frac{\partial u}{\partial t}(x_i, 0) = \frac{u(x_i, t_1) - u(x_i, 0)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \tilde{\mu}_i), \tag{12.23}$$

for some $\tilde{\mu}_i$ in $(0, t_1)$. Solving for $u(x_i, t_1)$ in the equation gives

$$\begin{aligned} u(x_i, t_1) &= u(x_i, 0) + k\frac{\partial u}{\partial t}(x_i, 0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \tilde{\mu}_i) \\ &= u(x_i, 0) + kg(x_i) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \tilde{\mu}_i). \end{aligned}$$

Deleting the truncation term gives the approximation,

$$w_{i,1} = w_{i,0} + kg(x_i), \quad \text{for each } i = 1, \ldots, m-1. \tag{12.24}$$

However, this approximation has truncation error of only $O(k)$ whereas the truncation error in Eq. (12.19) is $O(k^2)$.

Improving the Initial Approximation

To obtain a better approximation to $u(x_i, 0)$, expand $u(x_i, t_1)$ in a second Maclaurin polynomial in t. Then

$$u(x_i, t_1) = u(x_i, 0) + k\frac{\partial u}{\partial t}(x_i, 0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, 0) + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i, \hat{\mu}_i),$$

for some $\hat{\mu}_i$ in $(0, t_1)$. If f'' exists, then

$$\frac{\partial^2 u}{\partial t^2}(x_i, 0) = \alpha^2\frac{\partial^2 u}{\partial x^2}(x_i, 0) = \alpha^2\frac{d^2 f}{dx^2}(x_i) = \alpha^2 f''(x_i)$$

and

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{\alpha^2 k^2}{2}f''(x_i) + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i, \hat{\mu}_i).$$

This produces an approximation with error $O(k^3)$:

$$w_{i1} = w_{i0} + kg(x_i) + \frac{\alpha^2 k^2}{2}f''(x_i).$$

$f \in C^4[0, 1]$ but $f''(x_i)$ is not readily available, we can use the difference equation in Eq. (4.9) to write

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} - \frac{h^2}{12}f^{(4)}(\tilde{\xi}_i),$$

for some $\tilde{\xi}_i$ in (x_{i-1}, x_{i+1}). This implies that

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{k^2\alpha^2}{2h^2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] + O(k^3 + h^2k^2).$$

Because $\lambda = k\alpha/h$, we can write this as

$$\begin{aligned} u(x_i, t_1) &= u(x_i, 0) + kg(x_i) + \frac{\lambda^2}{2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] + O(k^3 + h^2k^2) \\ &= (1 - \lambda^2)f(x_i) + \frac{\lambda^2}{2}f(x_{i+1}) + \frac{\lambda^2}{2}f(x_{i-1}) + kg(x_i) + O(k^3 + h^2k^2). \end{aligned}$$

Thus, the difference equation,

$$w_{i,1} = (1-\lambda^2)f(x_i) + \frac{\lambda^2}{2}f(x_{i+1}) + \frac{\lambda^2}{2}f(x_{i-1}) + kg(x_i), \qquad (12.25)$$

can be used to find $w_{i,1}$, for each $i = 1, 2, \ldots, m-1$. To determine subsequent approximates we use the system in (12.22).

Algorithm 12.4 uses Eq. (12.25) to approximate $w_{i,1}$, although Eq. (12.24) could also be used. It is assumed that there is an upper bound for the value of t to be used in the stopping technique, and that $k = T/N$, where N is also given.

ALGORITHM 12.4

Wave Equation Finite-Difference

To approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < l, \quad 0 < t < T,$$

subject to the boundary conditions

$$u(0,t) = u(l,t) = 0, \quad 0 < t < T,$$

and the initial conditions

$$u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for} \quad 0 \le x \le l,$$

INPUT endpoint l; maximum time T; constant α; integers $m \ge 2$, $N \ge 2$.

OUTPUT approximations $w_{i,j}$ to $u(x_i, t_j)$ for each $i = 0, \ldots, m$ and $j = 0, \ldots, N$.

Step 1 Set $h = l/m$;
$k = T/N$;
$\lambda = k\alpha/h$.

Step 2 For $j = 1, \ldots, N$ set $w_{0,j} = 0$;
$w_{m,j} = 0$;

Step 3 Set $w_{0,0} = f(0)$;
$w_{m,0} = f(l)$.

Step 4 For $i = 1, \ldots, m-1$ (*Initialize for $t = 0$ and $t = k$.*)
set $w_{i,0} = f(ih)$;
$$w_{i,1} = (1-\lambda^2)f(ih) + \frac{\lambda^2}{2}[f((i+1)h) + f((i-1)h)] + kg(ih).$$

Step 5 For $j = 1, \ldots, N-1$ (*Perform matrix multiplication.*)
for $i = 1, \ldots, m-1$
set $w_{i,j+1} = 2(1-\lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}$.

Step 6 For $j = 0, \ldots, N$
set $t = jk$;
for $i = 0, \ldots, m$
set $x = ih$;
OUTPUT $(x, t, w_{i,j})$.

Step 7 STOP. (*The procedure is complete.*) ■

Example 1 Approximate the solution to the hyperbolic problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) - 4\frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad \text{for} \quad 0 < t,$$

and initial conditions

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1, \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le 1,$$

using $h = 0.1$ and $k = 0.05$. Compare the results with the exact solution

$$u(x,t) = \sin \pi x \cos 2\pi t.$$

Solution Choosing $h = 0.1$ and $k = 0.05$ gives $\lambda = 1$, $m = 10$, and $N = 20$. We will choose a maximum time $T = 1$ and apply the Finite-Difference Algorithm 12.4. This produces the approximations $w_{i,N}$ to $u(0.1i, 1)$ for $i = 0, 1, \ldots, 10$. These results are shown in Table 12.6 and are correct to the places given. ■

Table 12.6

x_i	$w_{i,20}$
0.0	0.0000000000
0.1	0.3090169944
0.2	0.5877852523
0.3	0.8090169944
0.4	0.9510565163
0.5	1.0000000000
0.6	0.9510565163
0.7	0.8090169944
0.8	0.5877852523
0.9	0.3090169944
1.0	0.0000000000

The results of the example were very accurate, more so than the truncation error $O(k^2 + h^2)$ would lead us to believe. This is because the true solution to the equation is infinitely differentiable. When this is the case, Taylor series gives

$$\frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2}$$
$$= \frac{\partial^2 u}{\partial x^2}(x_i,t_j) + 2\left[\frac{h^2}{4!}\frac{\partial^4 u}{\partial x^4}(x_i,t_j) + \frac{h^4}{6!}\frac{\partial^6 u}{\partial x^6}(x_i,t_j) + \cdots\right]$$

and

$$\frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2}$$
$$= \frac{\partial^2 u}{\partial t^2}(x_i,t_j) + 2\left[\frac{k^2}{4!}\frac{\partial^4 u}{\partial t^4}(x_i,t_j) + \frac{h^4}{6!}\frac{\partial^6 u}{\partial t^6}(x_i,t_j) + \cdots\right].$$

Since $u(x,t)$ satisfies the partial differential equation,

$$\frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2} - \alpha^2\frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2}$$
$$= 2\left[\frac{1}{4!}\left(k^2\frac{\partial^4 u}{\partial t^4}(x_i,t_j) - \alpha^2 h^2\frac{\partial^4 u}{\partial x^4}(x_i,t_j)\right)\right.$$
$$\left. + \frac{1}{6!}\left(k^4\frac{\partial^6 u}{\partial t^6}(x_i,t_j) - \alpha^2 h^4\frac{\partial^6 u}{\partial x^6}(x_i,t_j)\right) + \cdots\right]. \qquad (12.26)$$

However, differentiating the wave equation gives

$$k^2\frac{\partial^4 u}{\partial t^4}(x_i,t_j) = k^2\frac{\partial^2}{\partial t^2}\left[\alpha^2\frac{\partial^2 u}{\partial x^2}(x_i,t_j)\right] = \alpha^2 k^2\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 u}{\partial t^2}(x_i,t_j)\right]$$
$$= \alpha^2 k^2\frac{\partial^2}{\partial x^2}\left[\alpha^2\frac{\partial^2 u}{\partial x^2}(x_i,t_j)\right] = \alpha^4 k^2\frac{\partial^4 u}{\partial x^4}(x_i,t_j),$$

and we see that since $\lambda^2 = (\alpha^2 k^2/h^2) = 1$, we have

$$\frac{1}{4!}\left[k^2\frac{\partial^4 u}{\partial t^4}(x_i,t_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(x_i,t_j)\right] = \frac{\alpha^2}{4!}[\alpha^2 k^2 - h^2]\frac{\partial^4 u}{\partial x^4}(x_i,t_j) = 0.$$

Continuing in this manner, all the terms on the right-hand side of (12.26) are 0, implying that the local truncation error is 0. The only errors in Example 1 are those due to the approximation of $w_{i,1}$ and to round-off.

As in the case of the Forward-Difference method for the heat equation, the Explicit Finite-Difference method for the wave equation has stability problems. In fact, it is necessary that $\lambda = \alpha k/h \le 1$ for the method to be stable. (See [IK], p. 489.) The explicit method given in Algorithm 12.4, with $\lambda \le 1$, is $O(h^2 + k^2)$ convergent if f and g are sufficiently differentiable. For verification of this, see [IK], p. 491.

Although we will not discuss them, there are implicit methods that are unconditionally stable. A discussion of these methods can be found in [Am], p. 199, [Mi], or [Sm,G].

EXERCISE SET 12.3

1. Approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \quad 0 < t;$$
$$u(0,t) = u(1,t) = 0, \quad 0 < t,$$
$$u(x,0) = \sin \pi x, \quad 0 \le x \le 1,$$
$$\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le 1,$$

using the Finite-Difference Algorithm 12.4 with $m = 4$, $N = 4$, and $T = 1.0$. Compare your results at $t = 1.0$ to the actual solution $u(x,t) = \cos \pi t \sin \pi x$.

2. Approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{16\pi^2}\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 0.5,\ 0 < t;$$
$$u(0,t) = u(0.5,t) = 0, \quad 0 < t,$$
$$u(x,0) = 0, \quad 0 \le x \le 0.5,$$
$$\frac{\partial u}{\partial t}(x,0) = \sin 4\pi x, \quad 0 \le x \le 0.5,$$

using the Finite-Difference Algorithm 12.4 with $m = 4$, $N = 4$ and $T = 0.5$. Compare your results at $t = 0.5$ to the actual solution $u(x,t) = \sin t \sin 4\pi x$.

3. Approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < \pi,\ 0 < t;$$
$$u(0,t) = u(\pi,t) = 0, \quad 0 < t,$$
$$u(x,0) = \sin x, \quad 0 \le x \le \pi,$$
$$\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le \pi,$$

using the Finite-Difference Algorithm with $h = \pi/10$ and $k = 0.05$, with $h = \pi/20$ and $k = 0.1$, and then with $h = \pi/20$ and $k = 0.05$. Compare your results at $t = 0.5$ to the actual solution $u(x,t) = \cos t \sin x$.

4. Repeat Exercise 3, using in Step 4 of Algorithm 12.4 the approximation

$$w_{i,1} = w_{i,0} + kg(x_i), \quad \text{for each } i = 1, \ldots, m-1.$$

5. Approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \ 0 < t;$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t,$$

$$u(x,0) = \sin 2\pi x, \quad 0 \le x \le 1,$$

$$\frac{\partial u}{\partial t}(x,0) = 2\pi \sin 2\pi x, \quad 0 \le x \le 1,$$

using Algorithm 12.4 with $h = 0.1$ and $k = 0.1$. Compare your results at $t = 0.3$ to the actual solution $u(x,t) = \sin 2\pi x(\cos 2\pi t + \sin 2\pi t)$.

6. Approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \ 0 < t;$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t,$$

$$u(x,0) = \begin{cases} 1, & 0 \le x \le \frac{1}{2}, \\ -1, & \frac{1}{2} < x \le 1, \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le 1.$$

using Algorithm 12.4 with $h = 0.1$ and $k = 0.1$.

7. The air pressure $p(x,t)$ in an organ pipe is governed by the wave equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \quad 0 < x < l, \ 0 < t,$$

where l is the length of the pipe, and c is a physical constant. If the pipe is open, the boundary conditions are given by

$$p(0,t) = p_0 \quad \text{and} \quad p(l,t) = p_0.$$

If the pipe is closed at the end where $x = l$, the boundary conditions are

$$p(0,t) = p_0 \quad \text{and} \quad \frac{\partial p}{\partial x}(l,t) = 0.$$

Assume that $c = 1$, $l = 1$, and the initial conditions are

$$p(x,0) = p_0 \cos 2\pi x, \quad \text{and} \quad \frac{\partial p}{\partial t}(x,0) = 0, \quad 0 \le x \le 1.$$

a. Approximate the pressure for an open pipe with $p_0 = 0.9$ at $x = \frac{1}{2}$ for $t = 0.5$ and $t = 1$, using Algorithm 12.4 with $h = k = 0.1$.

b. Modify Algorithm 12.4 for the closed-pipe problem with $p_0 = 0.9$, and approximate $p(0.5, 0.5)$ and $p(0.5, 1)$ using $h = k = 0.1$.

8. In an electric transmission line of length l that carries alternating current of high frequency (called a "lossless" line), the voltage V and current i are described by

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}, \quad 0 < x < l, \ 0 < t;$$

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}, \quad 0 < x < l, \ 0 < t;$$

where L is the inductance per unit length, and C is the capacitance per unit length. Suppose the line is 200 ft long and the constants C and L are given by

$$C = 0.1 \text{ farads/ft} \quad \text{and} \quad L = 0.3 \text{ henries/ft}.$$

Suppose the voltage and current also satisfy

$$V(0,t) = V(200,t) = 0, \quad 0 < t;$$
$$V(x,0) = 110\sin\frac{\pi x}{200}, \quad 0 \le x \le 200;$$
$$\frac{\partial V}{\partial t}(x,0) = 0, \quad 0 \le x \le 200;$$
$$i(0,t) = i(200,t) = 0, \quad 0 < t;$$
$$i(x,0) = 5.5\cos\frac{\pi x}{200}, \quad 0 \le x \le 200;$$

and

$$\frac{\partial i}{\partial t}(x,0) = 0, \quad 0 \le x \le 200.$$

Approximate the voltage and current at $t = 0.2$ and $t = 0.5$ using Algorithm 12.4 with $h = 10$ and $k = 0.1$.

12.4 An Introduction to the Finite-Element Method

Finite elements began in the 1950s in the aircraft industry. Use of the techniques followed a paper by Turner, Clough, Martin, and Topp [TCMT] that was published in 1956. Wide spread application of the methods required large computer recourses that were not available until the early 1970s.

The **Finite-Element method** is similar to the Rayleigh-Ritz method for approximating the solution to two-point boundary-value problems that was introduced in Section 11.5. It was originally developed for use in civil engineering, but it is now used for approximating the solutions to partial differential equations that arise in all areas of applied mathematics.

One advantage the Finite-Element method has over finite-difference methods is the relative ease with which the boundary conditions of the problem are handled. Many physical problems have boundary conditions involving derivatives and irregularly shaped boundaries. Boundary conditions of this type are difficult to handle using finite-difference techniques because each boundary condition involving a derivative must be approximated by a difference quotient at the grid points, and irregular shaping of the boundary makes placing the grid points difficult. The Finite-Element method includes the boundary conditions as integrals in a functional that is being minimized, so the construction procedure is independent of the particular boundary conditions of the problem.

In our discussion, we consider the partial differential equation

$$\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + r(x,y)u(x,y) = f(x,y), \tag{12.27}$$

with $(x,y) \in \mathcal{D}$, where $\mathcal{D}$ is a plane region with boundary $\mathcal{S}$.

Boundary conditions of the form

$$u(x,y) = g(x,y) \tag{12.28}$$