4.2 Eigenvalues and natural Frequencies $X \rightarrow$ Not required
غير مطلوبة

4.3 Free Vibration of undamped systems using Modal Analysis
↳ In book (Modal Analysis)

* Consider that we have a MDOF system with equation of motion

$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$

$[k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \quad \text{— (1)}$$

we were able to solve this equation using

$$\{x\} = \{X\} \sin(\omega t + \phi) \quad \text{—— (2)}$$

and we found that $\{X\}$ is a matrix, so $\Rightarrow [X] = [\{X^{(1)}\}\,\{X^{(n)}\}]$ ↳ modeshapes matrix
and we found $\omega_i \Rightarrow$ natural Frequencies
Now, we write equation (2) in form

$$\{x\} = [U]\, q_i(t) \quad \text{—— (3)}$$

$[U]$ is the modeshapes matrix $[X]$

time dependant function $\leftarrow$ $$q_i = \sin(\omega_i t + \phi_i) = A_i \cos\omega_i t + B_i \sin\omega_i t \quad \text{—(4)}$$

ω_i Natural Frequency of mode shape (i)
ϕ_i phase angle of mode shape (i)
A_i, B_i constants from Initial Conditions

If we substitute eq(3) in eq(1)

$$\Rightarrow [m][U]\ddot{q}_i + [k][U]q_i = \{0\} \quad \text{— (5)}$$

Pre-multiply equation (5) by $[U]^T$ ← transpose

$$\Rightarrow [U]^T[m][U]\ddot{q}_i + [U]^T[k][U]\,q_i = \{0\} \quad \text{—(6)}$$

Now,

$$[U]^T[m][U] = \begin{bmatrix} Mr_1 & 0 \\ 0 & Mr_2 \end{bmatrix} \quad \text{--- (7)}$$

$$[U]^T[k][U] = \begin{bmatrix} Kr_1 & 0 \\ 0 & Kr_2 \end{bmatrix}$$

At this step of the solution, we can normalize the mode shape matrix $[U]$ to be $[U_n]$, so that:

$[U_n]$: normalized modeshapes matrix

$$[U_n]^T[m][U_n] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I] \rightarrow \text{Identity matrix}$$

and

$$[U_n]^T[K][U_n] = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

, where ω_1 and ω_2 are natural Frequencies of the system

Now we can rewrite equation (6) to be

$$\underbrace{[U_n]^T[m][U_n]}_{=[I]}\ddot{q}_i + \underbrace{[U_n]^T[k][U_n]}_{[\omega_i^2]} q_i = \{0\}$$

$$\Rightarrow \quad \ddot{q}_i(t) + \omega_i^2 q_i(t) = 0 \quad , i = 1,2 \quad \text{--- (8)}$$

these are two ordinary differential equations that can be easily solved

If we have initial conditions

$$\{x(0)\} = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} \quad \text{and} \quad \{\dot{x}(0)\} = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} v_{10} \\ v_{20} \end{Bmatrix}$$

we can find A_i and B_i of equation (4)

Re consider eq (3)

$$\{x(t)\} = [U]\, q_i(t)$$

or using normalized modeshape matrix $[U_n]$

$$\{x(t)\} = [U_n]\, q_i(t)$$ ← premultiply by $[U_n]^T [m]$

$$\Rightarrow \quad [U_n]^T [m] \{x(t)\} = \underbrace{[U_n]^T [m] [U_n]}_{[I]}\, q_i(t)$$

$$\Rightarrow \quad q_i(t) = [U_n]^T [m] \{x(t)\} \qquad (9)$$

and derive with respect to time (t) (10)

$$\dot{q}_i(t) = [U_n]^T [m] \{\dot{x}(t)\}$$

Here we apply Intial conditions to find A_i and B_i

$$\Rightarrow \{x(t)\} = [U]\, q_i(t)$$

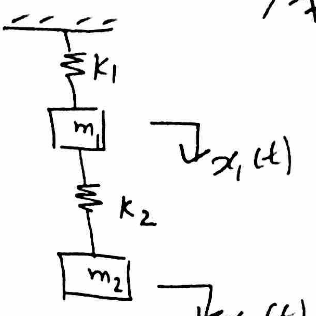
or

$$\{x(t)\} = [U_n]\, q_i(t)$$

For the 2DOF system shown, If the equation of motion is

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

where $[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$, $[k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$



$m_1 = 10\,kg$, $m_2 = 1\,Kg$

$K_1 = 30\,N/m$, $K_2 = 5\,N/m$

with intial conditions

$$\{x(0)\} = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ and, } \{\dot{x}(0)\} = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Find the final expressions of $x_1(t)$ and $x_2(t)$

Solution

* Before we find $x_1(t)$ and $x_2(t)$, we need to find natural Frequencies and mode shapes, as we did before

$$\{x(t)\} = \{X\} \sin(\omega t + \phi) \leftarrow \text{Substitute in EOM}$$

$$\Rightarrow \underbrace{([k] - \omega^2[m])}_{\det(\ )=0}\{X\} = 0$$

$$\det\left([k] - \omega^2[m]\right) = 0$$

$$\Rightarrow \det\begin{pmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{pmatrix} = \begin{pmatrix} 35-\omega^2 10 & -5 \\ -5 & 5-\omega^2 \end{pmatrix} = 10\omega^4 - 85\omega^2 - 150 = 0$$

solve $(10\omega^4 - 85\omega^2 - 150 = 0) \Rightarrow$

$\omega_1^2 = 2.5 \Rightarrow \omega_1 = 1.58$ rad/s

$\omega_2^2 = 6 \Rightarrow \omega_2 = 2.45$ rad/s

natural Frequencies

...ing natural Freq's, now we need to find modeshapes.

For $\omega_1^2 = 2.5$

$$([K] - \omega_1^2 [m])\{X^{(1)}\} = \{0\}$$

$$\begin{bmatrix} 10 & -5 \\ -5 & 2.5 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} 10X_1^{(1)} - 5X_2^{(1)} = 0 \\ -5X_1^{(1)} - 2.5X_2^{(1)} = 0 \end{matrix} \Rightarrow 2X_1^{(1)} = X_2^{(1)}$$

let $X_1^{(1)} = 1 \Rightarrow X_2^{(1)} = 2$

$$\Rightarrow \{X^{(1)}\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

For $\omega_2^2 = 6$

$$([K] - \omega_2^2 [m])\{X^{(2)}\} = \{0\}$$

$$\begin{bmatrix} -25 & -5 \\ -5 & -1 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} -25X_1^{(2)} - 5X_2^{(2)} = 0 \\ -5X_1^{(2)} - X_2^{(2)} = 0 \end{matrix} \Rightarrow 5X_1^{(2)} = -X_2^{(2)}$$

let $X_1^{(2)} = 1 \Rightarrow X_2^{(2)} = -5$

$$\Rightarrow \{X^{(2)}\} = \begin{Bmatrix} 1 \\ -5 \end{Bmatrix}$$

Now, we can find modeshapes matrix $[U]$

$$[U] = [\{X^{(1)}\} \ \{X^{(2)}\}] = \begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

After obtaing $[U]$, now we can find $[U_n]$ "Normalized modeshape matrix"

— Reconsider eq(7)

$$[U]^T [m] [U] = \begin{bmatrix} mr_1 & 0 \\ 0 & mr_2 \end{bmatrix} \Rightarrow [U] = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

the normalized modeshape matrix $[U_n]$, can be found as:

$$[U_n] = \begin{bmatrix} \dfrac{u_{11}}{\sqrt{mr_1}} & \dfrac{u_{12}}{\sqrt{mr_2}} \\ \dfrac{u_{21}}{\sqrt{mr_1}} & \dfrac{u_{22}}{\sqrt{mr_2}} \end{bmatrix}$$

— ur example ... int: required

$$[U]^T[m][U] = \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 35 \end{bmatrix}$$

mr_1, mr_2

$$\Rightarrow [U_n] = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{-5}{\sqrt{35}} \end{bmatrix} = \begin{bmatrix} 0.27 & 0.17 \\ 0.53 & -0.85 \end{bmatrix}$$

Now, we can easily find q_i and $\dot{q}_i$ for initial conditions. (Eq(9) and eq(10))

$$q_i(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = [U_n]^T[m]\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$

$q_1(t) = A_1\cos\omega_1 t + B_1\sin\omega_1 t$

$q_2(t) = A_2\cos\omega_2 t + B_2\sin\omega_2 t$

$$\dot{q}_i(t) = \begin{Bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{Bmatrix} = [U_n]^T[m]\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix}$$

$\dot{q}_1(t) = -\omega_1 A\sin\omega_1 t + \omega_1 B_1\cos\omega_1 t$

$\dot{q}_2(t) = -\omega_2 A_2\sin\omega_2 t + \omega_2 B_2\cos\omega_2 t$

Apply initial conditions $\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

$$\begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix} = \begin{bmatrix} 0.27 & 0.53 \\ 0.17 & -0.85 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2.67 \\ 1.69 \end{Bmatrix}$$

$$\Rightarrow q_1(0) = 2.67 = A_1\overset{1}{\cancel{\cos\omega_1(0)}} + B_1\overset{0}{\cancel{\sin\omega_1(0)}} \Rightarrow A_1 = 2.67$$

$$q_2(0) = 1.69 = A_2\overset{1}{\cancel{\cos\omega_2(0)}} + B_2\overset{0}{\cancel{\sin\omega_2(0)}} \Rightarrow A_2 = 1.69$$

For $\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\begin{Bmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \end{Bmatrix} = \begin{bmatrix} 0.27 & 0.53 \\ 0.17 & -0.85 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\dot{q}_1(0) = 0 = -\omega_1 A_1\overset{0}{\cancel{\sin\omega_1(0)}} + \omega_1 B_1\overset{1}{\cancel{\cos\omega_1(0)}} \Rightarrow B_1 = 0$$

$$\dot{q}_2(0) = 0 = -\omega_2 A_2\overset{0}{\cancel{\sin\omega_2(0)}} + \omega_2 B_2\cos\omega_2(0) \Rightarrow B_2 = 0$$

$$\Rightarrow q_1(t) = 2.67\cos\omega_1 t, \quad \omega_1 = 1.58 \text{ rad/s}$$

$$q_2(t) = 1.69\cos\omega_2 t, \quad \omega_2 = 2.45 \text{ rad/s}$$

...nally

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = [U_n] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{bmatrix} 0.27 & 0.17 \\ 0.53 & -0.85 \end{bmatrix} \begin{Bmatrix} 2.67 \cos \omega_1 t \\ 1.69 \cos \omega_2 t \end{Bmatrix}$$

$$\Rightarrow \quad x_1(t) = 0.72 \cos \omega_1 t + 0.12 \cos \omega_2 t$$

$$x_2(t) = 1.42 \cos \omega_1 t - 1.44 \cos \omega_2 t$$

It's a little lengthy but you can follow the procedure

* Please, let me know If you have any questions. *