*Chapter 29: Finite-Difference: Elliptic Equation

* Laplace Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

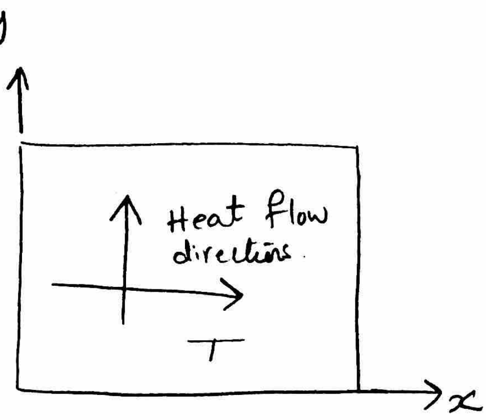


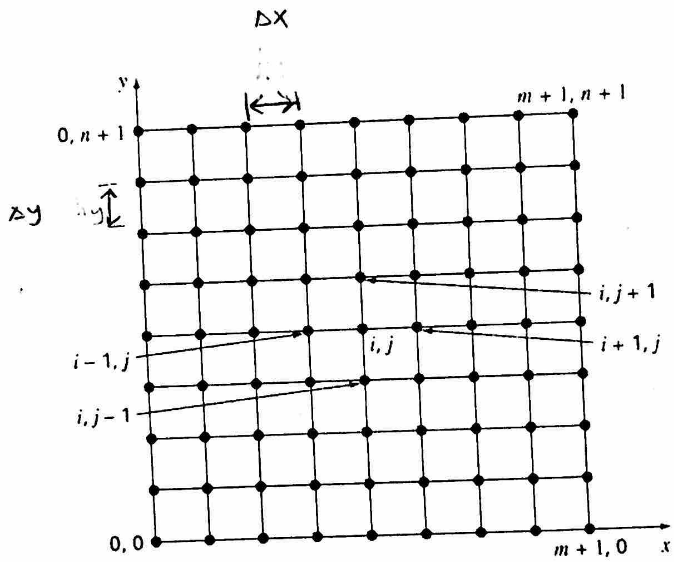
plate with Heat & temp Profiles

* Poisson's Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x,y)$$

Solution technique: Finite-Difference

⟹ Divide plate into mesh grid system (Discrete points)



* Using Centered finite differences

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$$

and

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

i: Counter x-direction
j: Counter y-direction

Δx and Δy are step size in x and y direction

Substitute into PDE

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

If $\Delta x = \Delta y$

$$\Rightarrow T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Laplace difference equation
$i, j = 0, 1, 2, \ldots, n$

* Boundary Conditions (BC's)

1- Fixed-type (Dirichlet Boundary conditions)

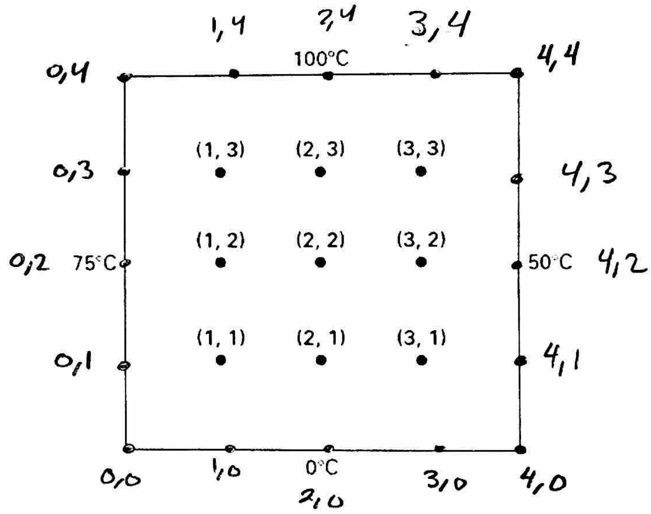
2- Derivative type. (Neumann Boundary condition)

1. Dirichlet BC's : plate BC's are held at constant temperature

Example plate with 3x3 internal mesh points (5x5 external)

we need to Find T_{ij} for internal points only using equation

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{ij} = 0$$



For point (1,1) [i=1, j=1]

$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$

But $T_{01} = 75$ and $T_{10} = 0$, thus:

$$\Rightarrow \quad -4T_{11} + T_{12} + T_{21} = -75 \qquad - Eq(1)$$

For point (3,3) [i=3, j=3]

$$T_{43} + T_{23} + T_{34} + T_{32} - 4T_{33} = 0$$

But $T_{43} = 50$ and $T_{34} = 100$, thus:

$$T_{23} + T_{32} - 4T_{33} = -150$$

If we do this for all internal points, we will get a system of nine linear Equations

Can be Solved Using any method of Ch 9 and Ch 10.

After performing finite Difference method of all 9 points, we get :

$$
\begin{array}{ccccccccccc}
4T_{11} & -T_{21} & & -T_{12} & & & & & & = & 75 \\
-T_{11} & +4T_{21} & -T_{31} & & -T_{22} & & & & & = & 0 \\
& -T_{21} & +4T_{31} & & & -T_{32} & & & & = & 50 \\
-T_{11} & & & +4T_{12} & -T_{22} & & -T_{13} & & & = & 75 \\
& -T_{21} & & -T_{12} & +4T_{22} & -T_{32} & & -T_{23} & & = & 0 \\
& & -T_{31} & & -T_{22} & +4T_{32} & & & -T_{33} & = & 50 \\
& & & -T_{12} & & & +4T_{13} & -T_{23} & & = & 175 \\
& & & & -T_{22} & & -T_{13} & +4T_{23} & -T_{33} & = & 100 \\
& & & & & -T_{32} & & -T_{23} & +4T_{33} & = & 150
\end{array}
$$

Can be Solved Using Gauss, Gauss-Jordan or LU.

2 Neumann BC's

- In this type of BC, one or more edge BC is a derivative (insulated edge). $\Rightarrow \frac{dT}{dx}=0$ or $\frac{dT}{dy}=0$
- For the previous example, If the plate left edge is insulated $\left(\frac{dT}{dx}=0\right)$. we can rewrite the finite difference equation.
- To Find $\Rightarrow$ left edge $(i=0)$, the Finite difference equation of of point's $(0,1)$, $(0,2)$ and $(0,3)$ can be written as: $(0,j)$

$$T_{1,j} + T_{-1,j} + T_{0,j+1} + T_{0,j-1} - 4T_{0j} = 0$$

$T_{-1,j}$ is Derivative type $\frac{dT}{dx}=0$

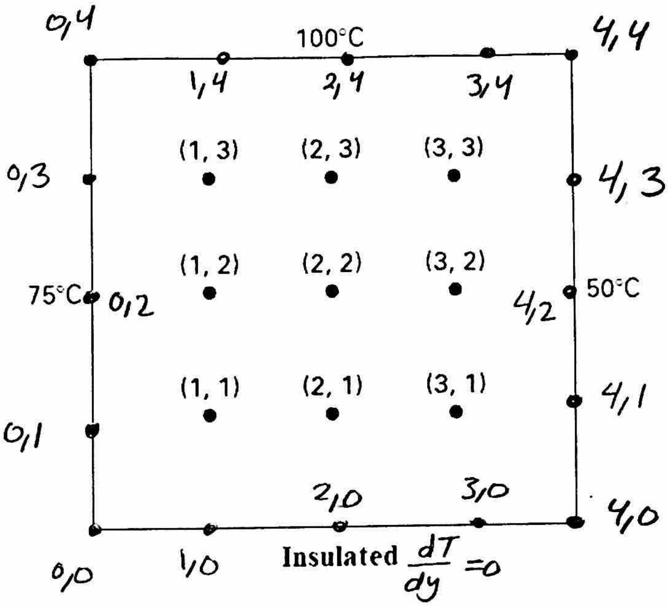
$$\frac{dT}{dx} = \frac{T_{1,j} - T_{-1,j}}{2\Delta x} \Rightarrow T_{-1,j} = T_{1,j} - 2\Delta x \frac{dT}{dx}$$

$$\Rightarrow \boxed{2T_{1,j} - 2\Delta x \frac{dT}{dx} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0}$$

Example, For the previous example, Solve if lower edge is insulated $\frac{dT}{dy}=0$

(4)

* we need To Find equations for (1,0), (2,0), (3,0) and (i,j), i=1,2,3 and j=1,2,3



* Remember $T_{0,j} = 75\,°C$ (left)

$T_{i,4} = 100°\,C$ (upper)

$T_{4,j} = 50°\,C$ (right)

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{ij} = 0$$

* For point (1,0), (i=1 j=0)

$$T_{20} + T_{00} + T_{11} + T_{1,-1} - 4T_{10} = 0$$

$T_{00} = 75°C$, $T_{1,-1}$ = derivative

$$\frac{dT}{dy} = \frac{T_{i,1} - T_{i,-1}}{2\Delta y}, \; i=1 \Rightarrow \frac{dT}{dy} = \frac{T_{1,1} - T_{1,-1}}{2\Delta y}$$

$$\Longrightarrow T_{1,-1} = T_{11} - 2\Delta y\frac{dT}{dy}, \quad \frac{dT}{dy}=0 \Rightarrow T_{1,-1} = T_{11}$$

Thus

$$T_{20} + 75 + 2T_{11} - 4T_{10} = 0 \Rightarrow T_{20} + 2T_{11} - 4T_{10} = -75$$

* If we do this for all points, we will get a system of 12 equations

* Remember we now have 3 more equations for points @ lower edge (1,0), (2,0) and (3,0)

* In Matrix Form

$$[A]_{12\times12}\{T\}_{12\times1} = \{C\}_{12\times1}$$

After finding equations for each point (12 points),
we get

$$
\begin{bmatrix}
4 & -1 & & -2 & & & & & & & & \\
-1 & 4 & -1 & & -2 & & & & & & & \\
 & -1 & 4 & & & -2 & & & & & & \\
-1 & & & 4 & -1 & & -1 & & & & & \\
 & -1 & & -1 & 4 & -1 & & -1 & & & & \\
 & & -1 & & -1 & 4 & -1 & & -1 & & & \\
 & & & -1 & & & 4 & -1 & & -1 & & \\
 & & & & -1 & & -1 & 4 & -1 & & -1 & \\
 & & & & & -1 & & -1 & 4 & & & -1 \\
 & & & & & & -1 & & & 4 & -1 & \\
 & & & & & & & -1 & & -1 & 4 & -1 \\
 & & & & & & & & -1 & & -1 & 4
\end{bmatrix}_{12\times 12}
\begin{Bmatrix} T_{10} \\ T_{20} \\ T_{30} \\ T_{11} \\ T_{21} \\ T_{31} \\ T_{12} \\ T_{22} \\ T_{32} \\ T_{13} \\ T_{23} \\ T_{33} \end{Bmatrix}_{12\times 1}
=
\begin{Bmatrix} 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 75 \\ 0 \\ 50 \\ 175 \\ 100 \\ 150 \end{Bmatrix}_{12\times 1}
$$

So lution $\Rightarrow$

$$
\begin{array}{lll}
T_{10} = 71.91 & T_{20} = 67.01 & T_{30} = 59.54 \\
T_{11} = 72.81 & T_{21} = 68.31 & T_{31} = 60.57 \\
T_{12} = 76.01 & T_{22} = 72.84 & T_{32} = 64.42 \\
T_{13} = 83.41 & T_{23} = 82.63 & T_{33} = 74.26
\end{array}
$$