2.4.3 Principal Stresses and Directions

For a stress state $[\sigma]$, we can find the principal values σ_i $(i=1,2,3)$ and associated principal directions $\{V_i\}$ $(i=1,2,3)$ as:

$$[\sigma]\{V_i\} = \sigma_i\{V_i\} \quad , \quad [\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

to find σ_i and $\{v_i\}$

$$[\sigma]\{v_i\} - \sigma_i\{v_i\} = 0 \Rightarrow \underbrace{\left([\sigma] - \sigma_i[I]\right)}_{\det[\,]=0}\underbrace{\{V_i\}}_{\neq 0} = 0$$

$$\left([\sigma] - \sigma_i[I]\right) = \begin{bmatrix} \sigma_{xx}-\sigma_i & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_i & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_i \end{bmatrix}$$

Principal equation

$$\det(\;) = 0 \Rightarrow \sigma_i^3 - I_1\sigma_i^2 + I_2\sigma_i - I_3 = 0$$

$\hookrightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3$ principal stresses

I_1, I_2 and I_3 are Stress Invariants

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = tr(\sigma)$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{yy}\sigma_{zz} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2$$

$$I_2 = \frac{1}{2}\left[tr^2(\sigma) - tr(\sigma^2)\right] , \quad [\sigma^2] = [\sigma][\sigma]$$

$$I_3 = \det([\sigma])$$

* Then we can find principal directions $\{V_1\}$, $\{V_2\}$ and $\{V_3\}$

* Then we find normalized directions $\{V_{n1}\}$, $\{V_{n2}\}$ and $\{V_{n3}\}$ as we learned previously.

Note

$$[Q^n] = \left[\, \{V_{n1}\}^T \,;\, \{V_{n2}\}^T \,;\, \{V_{n3}\}^T \,\right]$$

transformation matrix

$$[\sigma'] = [Q][\sigma][Q]^T = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

For $[\sigma']$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

Example Find principal stresses and principal directions

$$[\sigma] = \begin{bmatrix} -10 & 15 & 0 \\ 15 & 30 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution

① principal stresses

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

$$I_1 = tr(\sigma) = -10 + 30 + 0 = 20$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \cancel{\sigma_{xx}\sigma_{zz}}^{0} + \cancel{\sigma_{yy}\sigma_{zz}}^{0} - \sigma_{xy}^2 - \cancel{\sigma_{xz}^2}^{0} - \cancel{\sigma_{yz}^2}^{0}$$

$$= (-10)(30) - (15)^2 = -525$$

$$I_3 = det(\sigma) = 0$$

$$\Rightarrow \sigma^3 - 20\sigma^2 - 525\sigma = 0$$

$$\underset{\sigma=0}{\sigma}\,(\underbrace{\sigma^2 - 20\sigma - 525}_{=0}) = 0$$

$$\Rightarrow \sigma_1 = 35, \quad \sigma_2 = 0, \quad \sigma_3 = -15$$

② Principal directions $\quad [\sigma]\{V_i\} = \sigma_i\{V_i\}$

① $\{V_1\}$ from $\sigma_1 = 35$

$$[\sigma]\{V_1\} = \sigma_1\{V_1\}$$

$$\begin{bmatrix} -10 & 15 & 0 \\ 15 & 30 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} V_1^{(1)} \\ V_2^{(1)} \\ V_3^{(1)} \end{Bmatrix} = 35 \begin{Bmatrix} V_1^{(1)} \\ V_2^{(1)} \\ V_3^{(1)} \end{Bmatrix}$$

$$-10\,V_1^{(1)} + 15\,V_2^{(1)} = 35\,V_1^{(1)} \Rightarrow \boxed{V_2^{(1)} = 3\,V_1^{(1)}} \quad \text{— Eq(1)}$$

$$15\,V_1^{(1)} + 30\,V_2^{(1)} = 35\,V_2^{(1)} \Rightarrow \boxed{V_2^{(1)} = 3\,V_1^{(1)}} \quad \text{— Eq(2)}$$

$$0 + 0 + 0 = 35\,V_3^{(1)} \Rightarrow \boxed{V_3^{(1)} = 0} \quad \text{— Eq(3)}$$

let $V_1^{(1)} = 1 \Rightarrow V_2^{(1)} = 3$

$$\Rightarrow \{V_1\} = \begin{Bmatrix} 1 \\ 3 \\ 0 \end{Bmatrix}, \quad \vec{V_1} = \hat{i} + 3\hat{j}$$

Normalize $\quad \vec{V_{n1}} = \dfrac{1}{\sqrt{1^2+3^2}}\hat{i} + \dfrac{3}{\sqrt{1^2+3^2}}\hat{j}$

$$\vec{V_{n1}} = \frac{1}{\sqrt{10}}\hat{i} + \frac{3}{\sqrt{10}}\hat{j}$$

$$\{V_{n1}\} = \begin{Bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \\ 0 \end{Bmatrix}$$

② $\{V_2\}$ from $\sigma_2 = 0$

$$[\sigma]\{V_2\} = \sigma_2\{V_2\} = \begin{bmatrix} -10 & 15 & 0 \\ 15 & 30 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} V_1^{(2)} \\ V_2^{(2)} \\ V_3^{(2)} \end{Bmatrix} = 0 \begin{Bmatrix} V_1^{(2)} \\ V_2^{(2)} \\ V_3^{(2)} \end{Bmatrix} = 0$$

$$\left.\begin{aligned} -10 V_1^{(2)} + 15 V_2^{(2)} = 0 &\Rightarrow V_1^{(2)} = \frac{3}{2} V_2^{(2)} \quad - Eq(1) \\ 15 V_1^{(2)} + 30 V_2^{(2)} = 0 &\Rightarrow V_1^{(2)} = 2 V_2^{(2)} \quad - Eq(2) \end{aligned}\right\} \text{Solve}$$

$$\Rightarrow V_1^{(2)} = V_2^{(2)} = 0$$

we assume $V_3^{(2)} = 2$

$$\Rightarrow \{V_2\} = \begin{Bmatrix} 0 \\ 0 \\ 2 \end{Bmatrix}, \quad \vec{V_2} = 2\hat{k}$$

Normalize $\Rightarrow$ $\vec{V_{n2}} = \frac{2}{\sqrt{2^2}}\hat{k} = 1\hat{k}$

$$\Rightarrow \{V_{n2}\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

③ $\{V_3\}$ from $\sigma_3 = -15$

$$[\sigma]\{V_3\} = \sigma_3\{V_3\} \Rightarrow \begin{bmatrix} -10 & 15 & 0 \\ 15 & 30 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} V_1^{(3)} \\ V_2^{(3)} \\ V_3^{(3)} \end{Bmatrix} = -15 \begin{Bmatrix} V_1^{(3)} \\ V_2^{(3)} \\ V_3^{(3)} \end{Bmatrix}$$

$$-10 V_1^{(3)} + 15 V_2^{(3)} = -15 V_1^{(3)} \Rightarrow 15 V_2^{(3)} = -5 V_1^{(3)} \Rightarrow \boxed{V_1^{(3)} = -3 V_2^{(3)}}$$

$$15 V_1^{(3)} + 30 V_2^{(3)} = -15 V_2^{(3)} \Rightarrow V_1^{(3)} + 2V_2^{(3)} = -V_2^{(3)} \Rightarrow \boxed{V_1^{(3)} = -3 V_2^{(3)}}$$

$$0 = -15 V_3^{(3)} \Rightarrow \boxed{V_3^{(3)} = 0}$$

let $V_2^{(3)} = 1 \Rightarrow V_1^{(3)} = -3 \Rightarrow \{V_3\} = \begin{Bmatrix} -3 \\ 1 \\ 0 \end{Bmatrix}$

$\vec{V_3} = -3\hat{i} + j$

Normalize $\Rightarrow$ $\vec{V_{n3}} = \frac{-3}{\sqrt{(-3)^2+1^2}}\hat{i} + \frac{1}{\sqrt{(-3)^2+1^2}}\hat{j}$

$$= \frac{-3}{\sqrt{10}}\hat{i} + \frac{1}{\sqrt{10}}\hat{j}$$

$$\{V_{n3}\} = \begin{Bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \\ 0 \end{Bmatrix}$$

Example

$$[\sigma] = \begin{bmatrix} 20 & 30 & -10 \\ 30 & 10 & 80 \\ -10 & 80 & \sigma_{zz} \end{bmatrix} , \quad I_2 = -7800$$

Find σ_{zz}

Solution

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{xx}\sigma_{zz} + \sigma_{zz}\sigma_{yy} - \sigma_{xy}^2 - \sigma_{xz}^2 - \sigma_{yz}^2$$

$$-7{,}800 = (20)(10) + (20)(\sigma_{zz}) + (10)(\sigma_{zz}) - (30)^2 - (-10)^2 - (80)^2$$

$$\Rightarrow \boxed{\sigma_{zz} = -20}$$