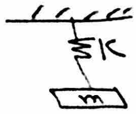
Review chapter one and Chapter two

* Chapter one

→ Free vibration (No Forces)

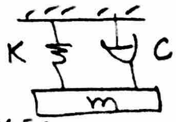
→ Undamped

$$m\ddot{x} + kx = 0$$
$$\ddot{x} + \omega_n^2 x = 0$$



→ Damped

$$m\ddot{x} + c\dot{x} + kx = 0$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$



Homogeneous ODE

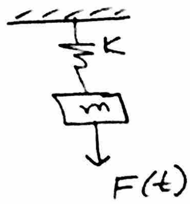
$$x(t) = Ce^{\lambda t}$$

* Chapter Two

→ Forced Vibration (with Forces)

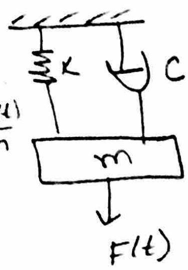
→ Undamped

$$m\ddot{x} + kx = F(t)$$
$$\ddot{x} + \omega_n^2 x = \frac{F(t)}{m}$$



→ Damped

$$m\ddot{x} + c\dot{x} + kx = F(t)$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$



Non-Homogeneous ODE

$$x(t) = x_h(t) + x_p(t)$$

→ particular

Homog.

→ Same as chapter 1

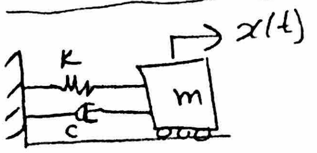
*Multidegrees of Freedom Systems (MDOF) – Chapter 4

Tuesday (24/03/2020)
and Thursday (26/03/2020)

* In multidegree-of-freedom systems (MDOF) we need more than one variable to describe the full motion of the system

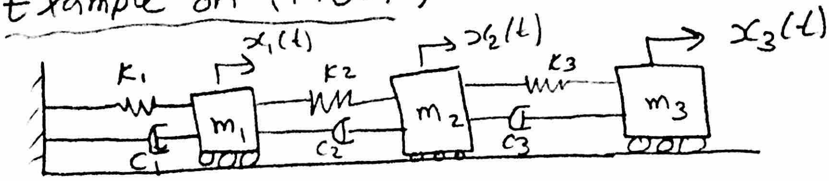
Remember: In single-degree-of-freedom (SDOF) systems we needed only one variable to describe the full motion of the system.

- Example on (SDOF)



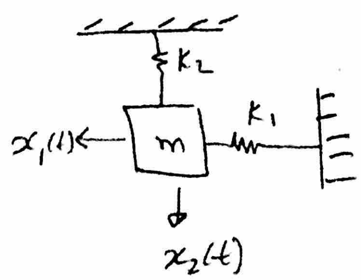
, In this system, we need only one variable to describe the motion of the system ($x(t)$)

- Example on (MDOF)



In the system above we need 3 variables to describe full motion of the system ($x_1(t)$, $x_2(t)$ and $x_3(t)$).
This is a 3 degrees-of-freedom-system

Another Example MDOF

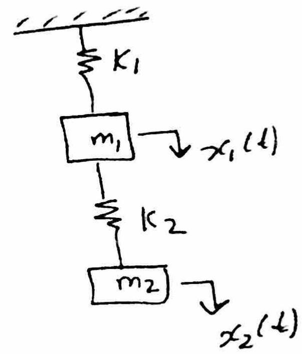


$x_1(t)$ and $x_2(t)$ are the degrees of freedom here

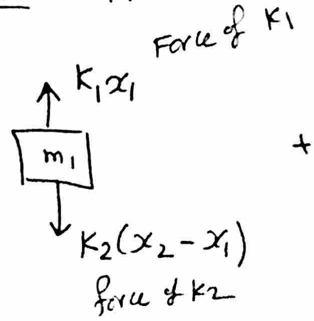
(this is a 2 degree of freedom system)

Two degrees of freedom systems. (undamped) and (unforced)

* To find the equation of motion of this system, we need to draw Free-body-diagram of m_1 and m_2



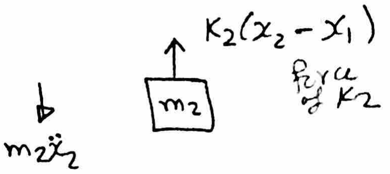
F.B.D (m_1)



$$+\downarrow \Sigma F = m_1 \ddot{x}_1 \Rightarrow m_1 \ddot{x}_1 = -K_1 x_1 + K_2(x_2 - x_1)$$

$$Eq(1) \Rightarrow m_1 \ddot{x}_1 + (K_1 + K_2)x_1 - K_2 x_2 = 0$$

F.B.D (m_2)



$$+\downarrow \Sigma F = m_2 \ddot{x}_2 \Rightarrow m_2 \ddot{x}_2 = -K_2(x_2 - x_1)$$

$$Eq(2) \Rightarrow m_2 \ddot{x} - K_2 x_1 + K_2 x_2 = 0$$

The equations of motion are:

$$m_1 \ddot{x}_1 + (K_1 + K_2)x_1 - K_2 x_2 = 0 \qquad -(1)$$

$$m_2 \ddot{x}_2 - K_2 x_1 + K_2 x_2 = 0 \qquad -(2)$$

Matrix Form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underbrace{[m]}_{\text{mass Matrix}} \underbrace{\{\ddot{x}\}}_{\text{Acceleration vector}} + \underbrace{[K]}_{\text{Stiffness Matrix}} \underbrace{\{x\}}_{\text{Displacement vector}} = \{0\}$$

{ } → vector

[ ] → matrix

2DF, cont'd

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$ — EOM : Equation of motion

To solve this system,

$x_1(t) = X_1 \sin(wt + \phi)$ (amplitude and phase)
$x_2(t) = X_2 \sin(wt + \phi)$ (" and ")

or, in matrix form

$\{x(t)\} = \{X\} \sin(wt + \phi)$, $\{x(t)\} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$, $\{X\} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$

$\{\dot{x}(t)\} = w\{X\} \cos(wt + \phi)$, $\{\dot{x}(t)\} = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix}$ (velocity)

$\{\ddot{x}(t)\} = -w^2 \{X\} \sin(wt + \phi)$, $\{\ddot{x}(t)\} = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix}$ (accelaration)

Substitute in EOM

$$-w^2 [m]\{X\} \sin(w\cancel{t} + \phi) + [k]\{X\} \sin(w\cancel{t} + \phi) = 0$$

$$\Rightarrow \left([k] - w^2[m]\right)\{X\} = 0 \quad \text{—— (A)}$$

المحددة

For non-trivial Solution ($\{X\} \neq 0$), the determinant of $([k] - w^2[m])$ should be equal to zero

$$\det\left([k] - w^2[m]\right) = 0$$

$$\Rightarrow \det \begin{bmatrix} -w^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -w^2 m_2 + k_2 \end{bmatrix} = 0$$

$$\Rightarrow m_1 m_2 w^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2) w^2 + k_1 k_2 = 0$$

This equation is called the (Characteristic Equation)

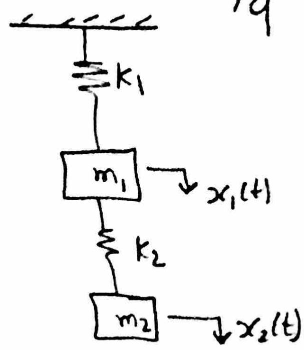
These are natural Frequencies $\Leftarrow$ The Solution of this equation finds ... $(w_1 \leq w_2)$

.ample

Find ω_1 and ω_2 of the following system

If $m_1 = 9\,kg$, $m_2 = 1\,kg$

$k_1 = 24\,N/m$, $k_2 = 3\,N/m$



Solution

EOM

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

where

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [k] = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$

Now, we need to find $\det([k] - \omega^2[m]) = 0$

$$\det \begin{bmatrix} 27-\omega^2 9 & -3 \\ -3 & 3-\omega^2 \end{bmatrix} = \omega^4 - 6\omega^2 + 8 = 0$$

Remember

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve

$$\omega^4 - 6\omega^2 + 8 = 0$$

let $\lambda^2 = \omega^4 \Rightarrow \omega^2 = \lambda$

$\omega = \sqrt{\lambda}$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$\lambda_{1,2} = \frac{+6 \pm \sqrt{36 - (4)(1)(8)}}{2} = \frac{6 \pm \sqrt{36-32}}{2} = \frac{6 \pm \sqrt{4}}{2}$$

$$\Rightarrow \lambda_1 = \frac{6-2}{2} \Rightarrow \lambda_1 = 2, \quad \lambda_2 = \frac{6+2}{2} \Rightarrow \lambda_2 = 4$$

$\Rightarrow \omega_1^2 = \lambda_1 = 2 \Rightarrow \omega_1 = \sqrt{2}$ rad/s

$\omega_2^2 = \lambda_2 = 4 \Rightarrow \omega_2 = 2$ rad/s

we only take positive values.

$$\boxed{\begin{matrix} \omega_1 = \sqrt{2} \text{ rad/s} \\ \omega_2 = 2 \text{ rad/s} \end{matrix}}$$

example

For the previous example, Find X_1 and X_2

$\{x(t)\} = \{X\} \sin(\omega t + \phi)$

Solution

Remember Equation (A) in page 3

$$([k] - \omega^2 [m]) \{X\} = 0$$

$\omega_1^2 = 2 \; (\lambda_1)$

$$([k] - \omega_1^2 [m]) \{X^{(1)}\} = 0 \Rightarrow \begin{bmatrix} 27 - 9(2) & -3 \\ -3 & +3 - 2 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \boxed{X_1^{(1)} = \frac{1}{3} X_2^{(1)}} \text{ — } i \Rightarrow \text{Let } X_2^{(1)} = 1 \Rightarrow X_1^{(1)} = 1/3 \Rightarrow \{X^{(1)}\} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$\omega_2^2 = 4 \; (\lambda_2)$

$$([k] - \omega_2^2 [m]) \{X^{(2)}\} = 0 \Rightarrow \begin{bmatrix} 27 - 9(4) & -3 \\ -3 & 3 - 4 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \boxed{X_1^{(2)} = -\frac{1}{3} X_2^{(2)}} \text{ — } ii$$

let $X_2^{(2)} = 1 \Rightarrow X_1^{(2)} = -1/3$

$$\Rightarrow \{X^{(2)}\} = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

therefore

$$\{X\} = [\{X^{(1)}\} \; \{X^{(2)}\}] = \begin{bmatrix} 1/3 & -1/3 \\ 1 & 1 \end{bmatrix}$$

$\{X^{(1)}\}$ and $\{X^{(2)}\}$ are the <u>mode shapes</u> of the system

ω_1 and ω_2 are the <u>natural frequencies</u> of the system

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mple, For the previous example
Find $x_1(t)$ and $x_2(t)$ given initial conditions

$x_1(0) = x_{10}$, $x_2(0) = x_{20}$
$\dot{x}_1(0) = v_{10}$, $\dot{x}_2(0) = v_{20}$

Solution

$$\{x(t)\} = \{X\} \sin(\omega t + \phi)$$

But now

ω_1, ω_2 we know them

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{Bmatrix}$$

A_1, A_2, ϕ_1 and ϕ_2
From Initial conditions

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} \frac{1}{3} A_1 \sin(\sqrt{2} t + \phi_1) - \frac{1}{3} A_2 \sin(2t + \phi_2) \\ A_1 \sin(\sqrt{2} t + \phi_1) + A_2 \sin(2t + \phi_2) \end{bmatrix}$$

Now, we apply Initial conditions to find A_1, A_2
ϕ_1 and ϕ_2

See Example 4.1.7 for details
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