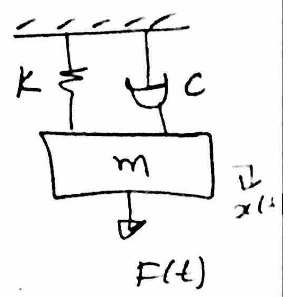
In this class we will discuss section 2.3 Alternative representation
In this section, there are three subsections (2.3.1, 2.3.2 and 2.3.3)
We will ONLY consider section 2.3.2 Complex response method and it is included in the exam. For sections 2.3.1 and 2.3.3, they are not required in the exam and will not be discussed.

2.3.2 Complex response method



* In this method, the excitation force F(t) in the shown Spring-Mass-damper system is assumed to be $F(t) = F_0 e^{i\omega t}$, where

F_0 is Force amplitude, ω: excitation frequency and $i = \sqrt{-1}$ is the complex number and (t) is the time.

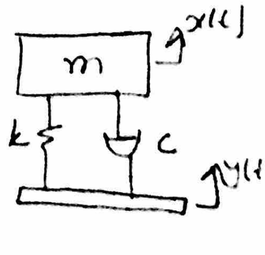
* For such kind of excitation the steady-state solution $x_p(t)$ (non-Homog. sol) is considered to be in the same Form of F(t), as:

$$x_p(t) = \underbrace{X}_{\text{Amplitude}} e^{i\overset{\text{excitation Freq}}{\omega} t}$$

$i = \sqrt{-1}$

t : time

* The same method can be used for base excitation problems in which $y(t) = \underbrace{Y}_{\text{Amplitude}} e^{i\omega t}$

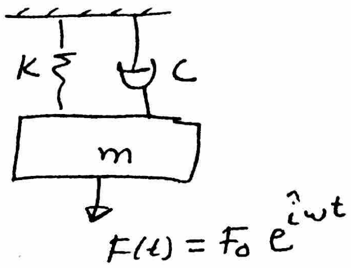


Here also the absolute motion $x_p(t) = X e^{i\omega t}$
and relative motion $z_p(t) = Z e^{i\omega t}$
As will be shown next!

2.3 Alternative representation

- 2.3.1 Graphical method ✗
- 2.3.2 Complex Response Method ✓
- 2.3.3 Transfer function method ✗

* Forced vibration



$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 e^{\hat{i}\omega t} \quad , \quad f_0 = \frac{F_0}{m}$$

$$x(t) = x_h(t) + x_p(t)$$

$$x_p(t) = X e^{\hat{i}\omega t}$$

$$\dot{x}_p(t) = \hat{i}\omega X e^{\hat{i}\omega t}$$

$$\ddot{x}_p(t) = -\omega^2 X e^{\hat{i}\omega t}$$

$$\Rightarrow -\omega^2 X \cancel{e^{\hat{i}\omega t}} + \hat{i}\, 2\zeta\omega\omega_n X \cancel{e^{\hat{i}\omega t}} + \omega_n^2 X \cancel{e^{\hat{i}\omega t}} = f_0 \cancel{e^{\hat{i}\omega t}}$$

$$X = \frac{f_0}{\omega_n^2 - \omega^2 + 2\hat{i}\zeta\omega\omega_n}$$

complex number
- magnitude
- phase

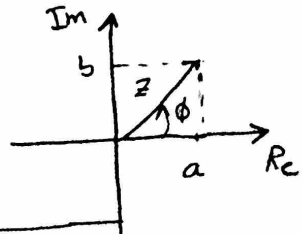
$$X = \underbrace{|X|}_{\text{mag.}} e^{i\phi \to \text{phase}}$$

$$z = a + \hat{i}b$$

a → real, b → Imag.

$$= |z|\, e^{i\phi}$$

|z| → mag., φ → phase



conjugate

$$X = \frac{f_0}{(\omega_n^2 - \omega^2) + 2\hat{i}\zeta\omega\omega_n} \cdot \frac{\omega_n^2 - \omega^2 - 2\hat{i}\zeta\omega\omega_n}{\omega_n^2 - \omega^2 - 2\hat{i}\zeta\omega\omega_n}$$

$$= \frac{f_0\left[(\omega_n^2 - \omega^2) - 2\hat{i}\zeta\omega\omega_n\right]}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

$$= \underbrace{\frac{f_0(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}}_{\text{Real}} - \underbrace{\frac{f_0(2\hat{i}\zeta\omega\omega_n)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}}_{\text{Imaginary}}$$

$$z = \sqrt{a^2 + b^2} = \sqrt{Re^2 + Im^2}$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\text{Imag.}}{\text{Real}}\right)$$

Conjugate of complex number

$$z = a + \hat{i}b \to \text{conjugate}(z) = a - \hat{i}b$$

magnitude $|X| = \sqrt{Re^2 + Im^2}$

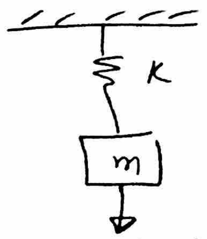
$$|X| = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}}$$

$$\phi = \tan^{-1}\left(\frac{Im}{Re}\right) = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)$$

Same as before using sin/cos

Undamped system

$\ddot{x} + \omega_n^2 x = f_0 e^{i\omega t}$ $\qquad f_0 = F_0/m$



$F(t) = F_0 e^{\hat{i}\omega t}$

$x(t) = x_h(t) + x_p(t)$

$x_p(t) = X e^{\hat{i}\omega t}$

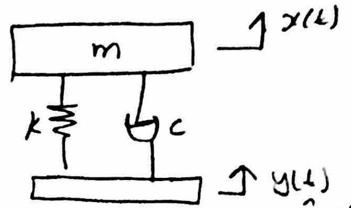
$\dot{x}_p(t) = \hat{i}\omega X e^{i\omega t}$

$\ddot{x}_p(t) = -\omega^2 X e^{i\omega t}$

$-\omega^2 X \cancel{e^{i\omega t}} + \omega_n^2 X \cancel{e^{i\omega t}} = f_0 \cancel{e^{i\omega t}}$

$\Rightarrow \boxed{X = \frac{f_0}{\omega_n^2-\omega^2}}$

* **Base Excitation** (absolute motion $x(t)$)



$y(t) = Y e^{i\omega_b t}$

$\dot{y} = i\omega_b Y e^{i\omega_b t}$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n \dot{y} + \omega_n^2 y$$

$$x(t) = x_h(t) + x_p(t)$$

$$x_p(t) = X e^{i\omega_b t}$$

$$\dot{x}_p(t) = i\omega_b X e^{i\omega_b t}$$

$$\ddot{x}_p(t) = -\omega_b^2 X e^{i\omega_b t}$$

$$\Rightarrow -\omega_b^2 X e^{i\omega_b t} + i2\zeta\omega_b\omega_n X e^{i\omega_b t} + \omega_n^2 X e^{i\omega_b t} = i2\zeta\omega_b\omega_n Y e^{i\omega_b t} + \omega_n^2 Y e^{i\omega_b t}$$

$$X = \frac{Y(\omega_n^2 + 2i\zeta\omega_b\omega_n)}{\omega_n^2 - \omega_b^2 + i2\zeta\omega_b\omega_n}$$, If we multiply by the conjugate with some mathematical formulation

$$X = |X| e^{i\phi}$$

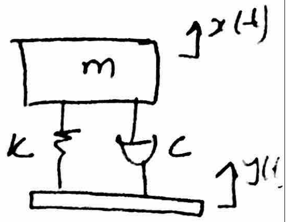
$$|X| = Y\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2} \Rightarrow \frac{|X|}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

Displacement transmissibility. (Same as before)

$$\phi = \tan^{-1}\left(\frac{2\zeta r^3}{1+(4\zeta^2-1)r^2}\right)$$

this ϕ includes θ_1 and θ_2
Using harmonic base motion
($y(t) = Y \sin \omega_b t$)

* Base Excitation (Relative motion)



$$y(t) = Y e^{i\omega_b t}$$
$$\dot{y}(t) = i\omega_b Y e^{i\omega_b t}$$
$$\ddot{y} = -\omega_b^2 Y e^{i\omega_b t}$$

Eq. of motion:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + K(x - y) = 0$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

Consider relative motion

$z(t) = x(t) - y(t)$, find $\dot{z} = \dot{x} - \dot{y}$ and $\ddot{z} = \ddot{x} - \ddot{y}$ and Substitute in Eq. of motion

$$\Rightarrow m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \qquad \div m \qquad (Eq(1))$$

$$\Rightarrow \ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = -\ddot{y} \qquad (Eq(2))$$

Solution to this equation

$$z(t) = z_h + \underline{\underline{z_p(t)}}$$

as before

$$z_p(t) = Z e^{i\omega_b t}, \quad \dot{z}_p(t) = i\omega_b Z e^{i\omega_b t}, \quad \ddot{z}_p(t) = -\omega_b^2 Z e^{i\omega t}$$

Substitute in eq(2)

$$\Rightarrow -\omega_b^2 Z \cancel{e^{i\omega_b t}} + i2\zeta\omega_n\omega_b Z \cancel{e^{i\omega_b t}} + \omega_n^2 Z \cancel{e^{i\omega t}} = +\omega_b^2 Y \cancel{e^{i\omega t}}$$

$$\Rightarrow Z = \frac{\omega_b^2 Y}{\omega_n^2 - \omega_b^2 + i2\zeta\omega_n\omega_b}$$

, Again If we multiply by complex Conjugate

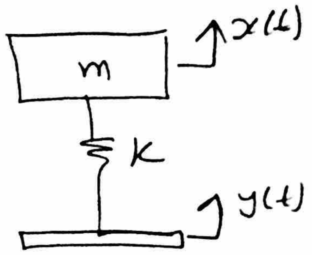
$$Z = \underset{Z_{mag.}}{|Z|}\, e^{i\eta} \leftarrow \text{phase}$$

$$\Rightarrow Z = \frac{\omega_b^2 Y}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}, \quad \eta = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right)$$

Same as we used $y(t) = Y \sin\omega_b t$

* Practice problem

Considering the undamped base excitation Problem shown, Find the steady state response (Absolute and relative) If $y(t) = Y e^{i\omega_b t}$



Hint: Steady-state response $x_p(t) = X e^{i\omega_b t}$ and $z_p(t) = Z_1 e^{i\omega_b t}$ and use the previously described procedure.