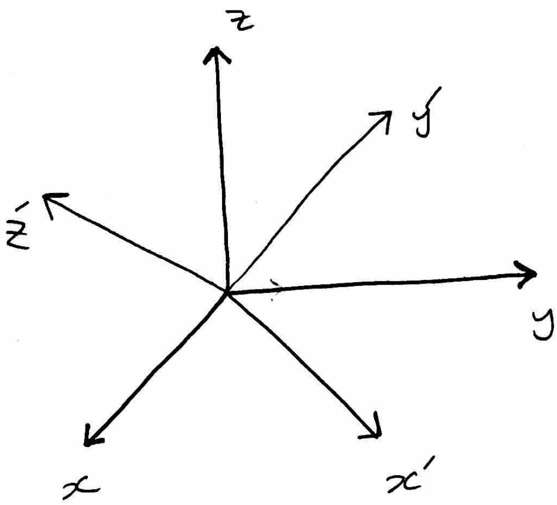
2.4 Transformation of stress, Principal stresses and other properties

- Transformation

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$



we need to find transformed stress state $[\sigma']$

$$[\sigma'] = [Q][\sigma][Q]^T \quad , \quad [\sigma'] = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{xy} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{xz} & \sigma'_{yz} & \sigma'_{zz} \end{bmatrix}$$

where $[Q]$ is the transformation matrix

$$[Q] = \begin{bmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{bmatrix}$$

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$$[Q] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

* Some properties of [Q]

① $[Q]^{-1} = [Q]^T \Rightarrow [Q]^{-1}[Q] = [Q][Q]^T = [I]$

②
$$\left.\begin{aligned} l_1^2 + l_2^2 + l_3^2 &= 1 \\ m_1^2 + m_2^2 + m_3^3 &= 1 \\ n_1^2 + n_2^2 + n_3^2 &= 1 \end{aligned}\right\} \quad \left.\begin{aligned} l_1 l_2 + m_1 m_2 + n_1 n_2 &= 0 \\ l_1 l_3 + m_1 m_3 + n_1 n_3 &= 0 \\ l_2 l_3 + m_2 m_3 + n_2 n_3 &= 0 \end{aligned}\right\} \quad \left\{\begin{aligned} & l_1^2 + m_1^2 + n_1^2 = 1 \\ & l_2^2 + m_2^2 + n_2^2 = 1 \\ & l_3^2 + m_3^3 + n_3^2 = 1 \\ & \Rightarrow l_i^2 + m_i^3 + n_i^2 = 1 \\ & (i = 1, 2, 3) \end{aligned}\right.$$

③
$$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0 \quad , \quad \sum_{i=1}^{3} l_i m_i = 0$$
$$l_1 n_1 + l_2 n_2 + l_3 n_3 = 0 \quad , \quad \sum_{i=1}^{3} l_i n_i = 0$$
$$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0 \quad , \quad \sum_{i=1}^{3} m_i n_i = 0$$

From $[\sigma'] = [Q][\sigma][Q]^T$

$$\sigma'_{xx} = l_1^2 \sigma_{xx} + m_1^2 \sigma_{yy} + n_1^2 \sigma_{zz} + 2 m_1 n_1 \sigma_{yz} + 2 n_1 l_1 \sigma_{xz} + 2 l_1 m_1 \sigma_{xy}$$

$$\sigma'_{yy} = l_2^2 \sigma_{xx} + m_2^2 \sigma_{yy} + n_2^2 \sigma_{zz} + 2 m_2 n_2 \sigma_{yz} + 2 n_2 l_2 \sigma_{xz} + 2 l_2 m_2 \sigma_{xy}$$

$$\sigma'_{zz} = l_3^2 \sigma_{xx} + m_3^2 \sigma_{yy} + n_3^2 \sigma_{zz} + 2 m_3 n_3 \sigma_{yz} + 2 n_3 l_3 \sigma_{xz} + 2 l_3 m_3 \sigma_{xy}$$

$$\sigma'_{xy} = l_1 l_2 \sigma_{xx} + m_1 m_2 \sigma_{yy} + n_1 n_2 \sigma_{zz} + (m_1 n_2 + m_2 n_1) \sigma_{yz} + (l_1 n_2 + l_2 n_1) \sigma_{xz} + (l_1 m_2 + l_2 m_1) \sigma_{xy}$$

$$\sigma'_{xz} = l_1 l_3 \sigma_{xx} + m_1 m_3 \sigma_{yy} + n_1 n_3 \sigma_{zz} + (m_1 n_3 + m_3 n_1) \sigma_{yz} + (l_1 n_3 + l_3 n_1) \sigma_{xz} + (l_1 m_3 + l_3 m_1) \sigma_{xy}$$

$$\sigma'_{yz} = l_2 l_3 \sigma_{xx} + m_2 m_3 \sigma_{yy} + n_2 n_3 \sigma_{zz} + (m_2 m_3 + m_3 n_2) \sigma_{yz} + (l_2 n_3 + l_3 n_2) \sigma_{xz} + (l_2 m_3 + l_3 m_2) \sigma_{xY}$$

If

$$\overrightarrow{\sigma'_x} = \sigma'_{xx}\,\hat{i} + \sigma'_{xy}\,\hat{j} + \sigma'_{xz}\,\hat{k}$$

$$\overrightarrow{\sigma'_y} = \sigma'_{xy}\,\hat{i} + \sigma'_{yy}\,\hat{j} + \sigma'_{yz}\,\hat{k}$$

$$\overrightarrow{\sigma'_z} = \sigma'_{xz}\,\hat{i} + \sigma'_{yz}\,\hat{j} + \sigma'_{zz}\,\hat{k}$$

and

$$\overrightarrow{N_1} = l_1\,\hat{i} + m_1\,\hat{j} + n_1\,\hat{k}$$

$$\overrightarrow{N_2} = l_2\,\hat{i} + m_2\,\hat{j} + n_2\,\hat{k}$$

$$\overrightarrow{N_3} = l_3\,\hat{i} + m_3\,\hat{j} + n_3\,\hat{k}$$

then

$$\sigma'_{xx} = \overrightarrow{\sigma'_x} \cdot \overrightarrow{N_1}$$

$$\sigma'_{yy} = \overrightarrow{\sigma'_y} \cdot \overrightarrow{N_2}$$

$$\sigma'_{zz} = \overrightarrow{\sigma'_z} \cdot \overrightarrow{N_3}$$

$$\sigma'_{xy} = \overrightarrow{\sigma'_x} \cdot \overrightarrow{N_2} \quad \text{or} \quad \sigma'_{yx} = \overrightarrow{\sigma'_y} \cdot \overrightarrow{N_1}$$

$$\sigma'_{xz} = \overrightarrow{\sigma'_x} \cdot \overrightarrow{N_3} \quad \text{or} \quad \sigma'_{zx} = \overrightarrow{\sigma'_z} \cdot \overrightarrow{N_1}$$

$$\overrightarrow{\sigma'_{yz}} = \overrightarrow{\sigma'_y} \cdot \overrightarrow{N_3} \quad \text{or} \quad \overrightarrow{\sigma'_{zy}} = \overrightarrow{\sigma'_z} \cdot N_2$$