

Chapter 27 Ordinary Differential Equations (ODEs)

Boundary-value and Eigen value problems

- What is the difference between Initial value problem (IVP) and Boundary-value problem (BVP)?
- Initial value problems, involves time. Initial values are always when x (or t) are equal to zero. Example:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \quad \text{Initial conditions} \quad \begin{array}{l} x(0) = x_0 \\ \dot{x}(0) = v_0 \\ \text{both at } t=0 \end{array}$$
$$\ddot{x} = \frac{d^2x}{dt^2}, \quad \dot{x} = \frac{dx}{dt}$$

- Boundary value problems. conditions are specified at extreme points or "Boundaries". like beams, heat transfer. Example

ODE

$$\frac{d^2T}{dx^2} + \alpha(T_a - T) = 0$$

Boundary conditions

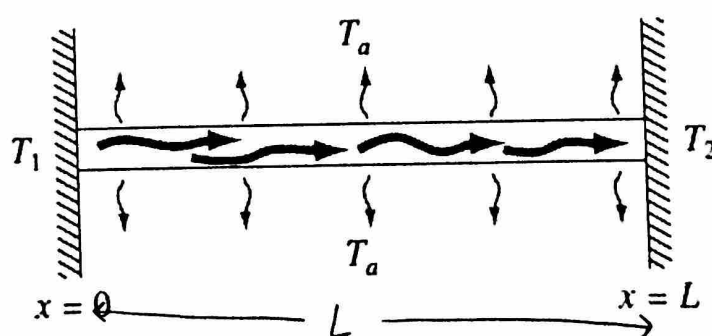
$$T(0) = T_1$$

$$T(L) = T_2$$

α : heat coefficient

T_a : Surrounding temperature

Heat transfer in a rod



27.1.1 The Shooting method

The shooting method is based on converting BVP to a system of differential equation with IVP.

Example

For the heat and temperature equation

$$\frac{d^2 T}{dx^2} + \alpha (T_a - T) = 0, \text{ If } l = 10 \text{ m}$$

$\alpha = 0.01 \text{ } 1/\text{m}^2$ and $T_a = 20^\circ\text{C}$. The boundary conditions are

$T(0) = 40^\circ\text{C}$, $T(10) = 200^\circ\text{C}$. Solve this equation

If step size is to be 2 m.

Solution

* We convert from BVP to IVP

$$\frac{dT}{dx} = z \quad \text{--- (1)}$$

$$\frac{d^2 T}{dx^2} = \frac{dz}{dx} = -\alpha (T_a - T) \quad \text{--- (2)}$$

The initial condition for equation (1):

$$T(0) = 40^\circ\text{C}$$

But we need an initial condition for eq (2) ($z(0) = ?$)

Remember, From equation (1),

$$\frac{dT}{dx} = z \Rightarrow dT = z dx \xrightarrow{\text{Integrate}} T = z x$$

$$\text{or } z(x) = \frac{T(x)}{x}, \quad z(0) = \frac{T(0)}{0} = \frac{40}{0} = \infty$$

needs to be given

Now, let's guess, for example

$$z(0) = 10$$

$$\Rightarrow \frac{dT}{dx} = z \quad , \quad T(0) = 40^\circ\text{C}$$

$$\frac{dz}{dx} = -\alpha(T_a - T) \quad , \quad z(0) = 10^\circ\text{C/m}$$

- Now, we solve this system of ODE's using RK4, (see 25.4),
Using $h=2$. Thus,

$$z_0 = z(0) = 10, \quad z_1 = z_0 + h, \dots, \quad z_5 = z_4 + h = z_0 + 5h = 10$$

we find $z_1, z_2, \dots, z_5$ and $T_1, T_2, \dots, T_5$

$$\text{at } x=10, \quad T(10) = 168.38^\circ\text{C}$$

- Remember from Boundary condition at $x=10$, $T(10) = 200^\circ\text{C}$

- we need another guess for $z(0) = ?$

Say, $z(0) = 20$, and resolve the system.

For this initial condition of z , $T(10) = 285.90^\circ\text{C}$

- Now,

$$z_1(0) = 10 \longrightarrow T_1(10) = 168.38^\circ\text{C}$$

$$z(0) = z_1^? \longrightarrow T_2(10) = 200^\circ\text{C}$$

$$z_2(0) = 20 \longrightarrow T_2(10) = 285.90^\circ\text{C}$$

Interpolation (Chapter 18)

$$\frac{z(0) - z_1(0)}{z_2(0) - z_1(0)} = \frac{T_2(10) - T_1(10)}{T_2(10) - T_1(10)}$$

$$z(0) = z_1 = z_1(0) + (z_2(0) - z_1(0)) \left(\frac{T_2(10) - T_1(10)}{T_2(10) - T_1(10)} \right)$$

substitute

$$Z = 10 + (20 - 10) \left(\frac{200 - 168.38}{285.90 - 168.38} \right)$$

$$Z = 12.70 \quad \text{or} \quad Z(0) = 12.70.$$

Now, we use this initial condition to solve the system of ODE's.

x	$T(x)$
0	$T(0) = 40$
2	$T(2) = 65.95$
4	$T(4) = 93.75$
6	$T(6) = 124.50$
8	$T(8) = 159.45$
10	$T(10) = 200.$

* Shooting method can be used to solve non-linear ODE's. However, smaller step size is required.

* In MATLAB, ode45 and ode23 functions can be used for solving systems of ODE's.

27.1.2 Finite-Difference method

For the equation,

$$\frac{d^2 T}{dx^2} + \alpha (T_a - T) = 0 \quad \text{--- (1)}$$

Remember, from chapter 23, the 2nd derivative using centered-finite-divided difference method is

$$\frac{d^2 T}{dx^2} = \frac{T(x_{i+1}) - 2T(x_i) + T(x_{i-1}))}{h^2}$$

Back in eq(1)

$$\frac{T(x_{i+1}) - 2T(x_i) + T(x_{i-1}))}{h^2} - \alpha (T(x_i) - T_a) = 0$$

$$i = 1, 2, 3 \dots$$

or,

$$-T(x_{i-1}) + (2 + \alpha h^2) T(x_i) - T(x_{i+1}) = \alpha h^2 T_a$$

$$-T_{i-1} + (2 + \alpha h^2) T_i - T_{i+1} = \alpha h^2 T_a$$

we will get set of linear equations that can be easily solved using

Gauss, Gauss-Jordan or LU method

Example Use finite divided difference approach to Sche

$$\frac{d^2 T}{dx^2} - \alpha (T - T_a) = 0$$

$$T(0) = 40 \quad T_a = 20$$

$$T(10) = 200, \alpha = 0.01$$

$$h = 2$$

$$-T_{i-1} + (2 + \alpha h^2) T_i - T_{i+1} = \alpha h^2 T_a$$

$$\underline{i = 1}$$

$$- \overset{40}{T_0} + \underset{0.01}{(2 + \underset{(2)^2}{\alpha h^2})} T_1 - T_2 = \underset{0.01}{\alpha} \underset{(2)^2}{h^2} \underset{(20)}{T_a}$$

$$-40 + 2.04 T_1 - T_2 = 0.8$$

$$\boxed{2.04 T_1 - T_2 = 40.8} \quad \text{--- ①}$$

$$T_0 = T(0)$$

$$T_1 = T(2)$$

$$T_2 = T(4)$$

$$T_3 = T(6)$$

$$T_4 = T(8)$$

$$T_5 = T(10)$$

$$\underline{i = 2}$$

$$-T_1 + (2 + \alpha h^2) T_2 - T_3 = \alpha h^2 T_a$$

$$\boxed{-T_1 + 2.04 T_2 - T_3 = 0.8} \quad \text{--- 2}$$

$$\underline{i = 3}$$

$$-T_2 + (2 + \alpha h^2) T_3 - T_4 = \alpha h^2 T_a$$

$$\boxed{-T_2 + 2.04 T_3 - T_4 = 0.8} \quad \text{--- 3}$$

$$\underline{i = 4}$$

$$-T_3 + (2 + \alpha h^2) T_4 - T_5 = \alpha h^2 T_a, \quad T_5 = T(10) = 200$$

$$\Rightarrow \boxed{-T_3 + 2.04 T_4 = 200.8} \quad \text{--- 4}$$

Matrix Form

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{Bmatrix}$$

Solve $T_1 = 65.97, T_2 = 93.78, T_3 = 124.54, T_4 = 159.48$

* Practice problem : Use Finite-divided difference method to solve for the Column Buckling Problem as defined as

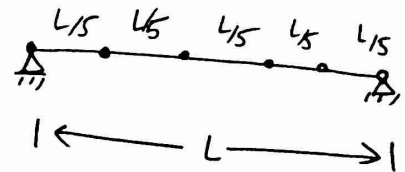
$$\frac{d^2 y}{dx^2} = \frac{M}{EI} \quad \circ \quad \begin{array}{l} M = \text{Coupling moment} \\ E = \text{Young's modulus} \\ I = \text{moment of Inertia} \end{array}$$

If $M = Py$, $P = \text{Force}$, $y = \text{location}$

$$\frac{d^2 y}{dx^2} = \frac{P y}{EI} \Rightarrow \left(\frac{d^2 y}{dx^2} - \frac{P y}{EI} = 0 \right)$$

If the Column length is L , Use stepsize

$$h = \frac{L}{5}$$



* All previously considered Boundary Conditions didn't include any derivatives - Dirichlet BC's.

* what if we have a derivative BC's ?
(Neuman's BC's).

For example

$$\frac{d^2 T}{dx^2} + \alpha (T_a - T) = 0 \quad ,$$

$$\frac{dT}{dx}(0) = T_a'$$

$$T(L) = T_b$$

Using Centered F.D.D

$$-T_{i-1} + (2 + \alpha h^2) T_i - T_{i+1} = \alpha h^2 T_a$$

But for 1st derivative using Centered F.D.D

$$\frac{dT}{dx} = \frac{T_{i+1} - T_{i-1}}{2h} \Rightarrow T_{i-1} = T_{i+1} - 2h \frac{dT}{dx}$$

Thus,

$$\Rightarrow (2 + \alpha h^2) T_i - 2 T_{i+1} = \alpha h^2 T_a - 2h \frac{dT}{dx}$$

This includes the effect of derivative type BC. in the solution!

27.2 Eigenvalue problems

①

- Examples
- ① Natural frequency and modeshapes (vibrations)
 - ② Buckling of columns. (Boundary-value problem)

① Vibration of multi-degrees of freedom system

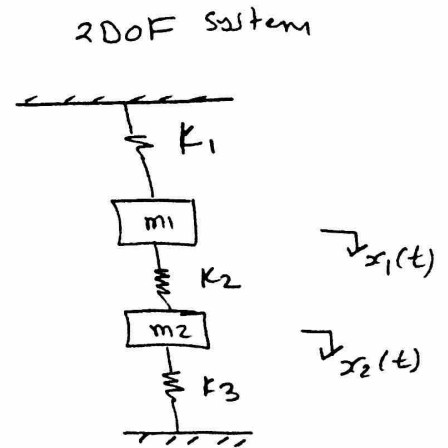
Equation of motion

$$m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = 0$$

$$\ddot{x} = \frac{d^2 x}{dt^2}$$

acceleration



B Solve

$$\{x\} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin \omega t$$

Substitute in eq. of motion

$$\ddot{x} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin \omega t$$

$$-\omega^2 m_1 A_1 \sin \omega t + (k_1 + k_2) A_1 \sin \omega t - k_2 A_2 \sin \omega t = 0$$

$$-\omega^2 m_2 A_2 \sin \omega t - k_2 A_1 \sin \omega t + (k_2 + k_3) A_2 \sin \omega t = 0$$

$$\Rightarrow \left[-\omega^2 m_1 + (k_1 + k_2) \right] A_1 - k_2 A_2 = 0$$

$$-k_2 A_1 + \left[-\omega^2 m_2 + k_2 + k_3 \right] A_2 = 0$$

matrix Form

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

det = 0

or = 0

Noway! $A_1 = A_2 = 0$
No solution

$\Rightarrow \det[\] = 0$
Eigenvalue
Problem

From $\det [] = 0 \rightarrow$ we get ω_1^2 and ω_2^2 (Eigenvalues), Nat Freq ②

and for $\omega_1^2 \xrightarrow{\text{Find}} \{A_1, A_2\}$
 $\omega_2^2 \xrightarrow{\text{Find}} \{A_1, A_2\} \rightarrow$ modeshapes $\rightarrow$ Eigenvectors!

For example $m_1 = m_2 = 40 \text{ kg}$, $k_1 = k_2 = k_3 = 200 \text{ N/m}$

$$\Rightarrow \det \Rightarrow \omega^4 - 20\omega^2 + 75 = 0$$

$$\text{let } \lambda = \omega^2 \Rightarrow \lambda^2 - 20\lambda + 75 = 0$$

$$\text{Solve } \lambda_1 = \omega_1^2 = 5 (\text{rad/s})^2, \quad \lambda_2 = \omega_2^2 = 15 (\text{rad/s})^2$$

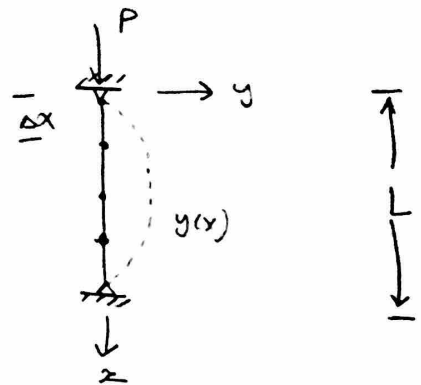
$$\text{For } \omega_1^2 = 5 \Rightarrow A_1 = -A_2$$

$$\omega_2^2 = 15 \Rightarrow A_1 = A_2$$

② Buckling of columns

Eq. of motion

$$\frac{d^2 y}{dx^2} + \rho^2 y = 0 \quad \text{, } \rho^2 = \frac{P}{EI}$$



Using Finite-difference "centered"

$$\frac{d^2 y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}, \quad \Delta x = \text{step size}$$

E : Elastic modulus

I : moment of Inertia

$$\Rightarrow y_{i-1} - (2 - (\Delta x)^2 \rho^2) y_i + y_{i+1} = 0$$

For 3 internal nodes ($\Delta x = \frac{L}{4}$) , L :

$i=1$

$$y_0 - (2 - (\Delta x)^2 \rho^2) y_1 + y_2 = 0$$

$\text{, } y_0 = \text{Boundary condition} = 0$

$i=2$

$$y_1 - (2 - (\Delta x)^2 \rho^2) y_2 + y_3 = 0$$

i=3

(3)

$$y_2 - (2 - (\Delta x)^2 \rho^2) y_3 + y_4 = 0$$

$$y_4 = \text{Boundary (condition)} = 0$$

matrix Form

$$\begin{bmatrix} 2 - (\Delta x)^2 \rho^2 & -1 & 0 \\ -1 & 2 - (\Delta x)^2 \rho^2 & -1 \\ 0 & -1 & 2 - (\Delta x)^2 \rho^2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\det [] = 0 \rightarrow \text{get } \rho^2 \text{ and equate to } \rho^2 = \frac{P}{EI}$$

Example If $E = 10 \times 10^9 \text{ Pa}$, $I = 1.25 \times 10^{-5} \text{ m}^4$, $L = 3 \text{ m}$

$$\Delta x = \frac{L}{4} = \frac{3}{4} = 0.75$$

$$\Rightarrow \begin{bmatrix} 2 - 0.5625 \rho^2 & -1 & 0 \\ -1 & 2 - 0.5625 \rho^2 & -1 \\ 0 & -1 & 2 - 0.5625 \rho^2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = 0$$

$$\det [] = 0$$

$$\rho_1 = \pm 1.0205$$

$$\rho_2 = \pm 1.8856$$

$$\rho_3 = \pm 2.4637$$