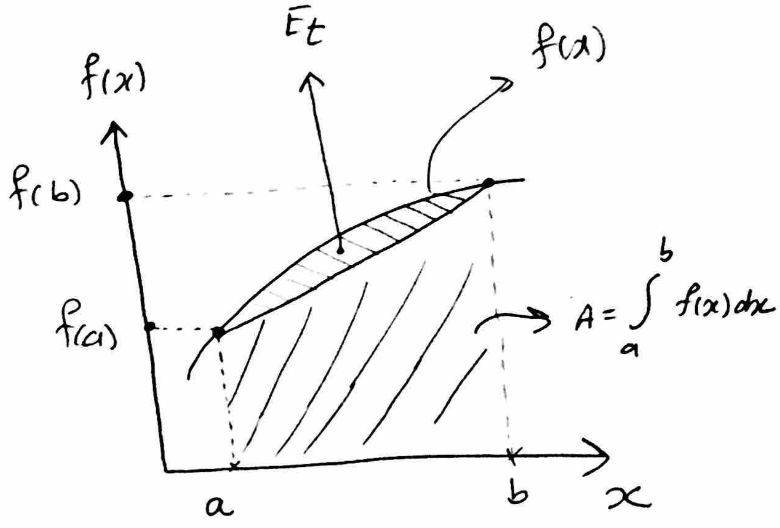
* Chapter 21: Newton-Cotes Integration Formulas
"Numerical Integration"

→ Trapezoidal Rule

→ Simpson's Rule → 1/3, → 3/8



21.1 Trapezoidal Rule

Integration (I) $\cong$ Area(A)

$$A = \frac{1}{2}\left(f(b) + f(a)\right)(b-a)$$

$$I \approx \frac{1}{2}(b-a)\left[f(b) + f(a)\right]$$

True error (E_t) = Exact - Approximate

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3 \quad , \quad a < \xi < b$$

"Derivations Box 21.2 textbook"

Approximate error (E_a)

$$E_a = -\frac{1}{12} \bar{f}''(x)(b-a)^3$$

↳ mean value of $f''(x)$

$$\bar{f}''(x) = \frac{\int_a^b f''(x)dx}{b-a}$$

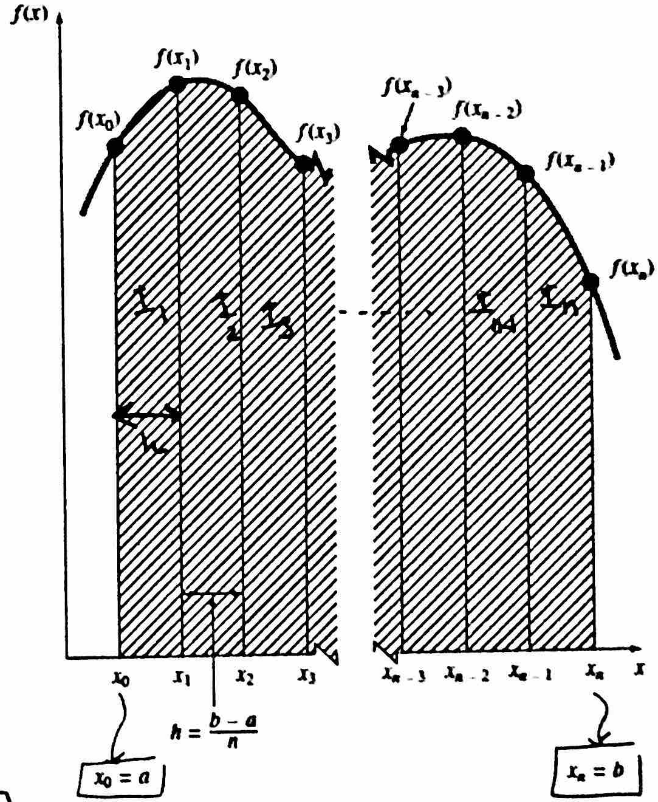
"Trapezoidal Rule always gives exact value of integration

If $f(x)$ → constant, → linear , why? $f''(x) = 0$ Always!

* How to improve the accuracy of Trapezoidal Rule?

↳ Take more trapezoidals

- For example, take "n" trapezoidals (n segments) of constant width (h), where $h = \dfrac{b-a}{n}$



$$I \cong I_1 + I_2 + \cdots + I_{n-1} + I_n$$

$$= \frac{1}{2}h\left(f(x_0)+f(x_1)\right) + \frac{1}{2}h\left(f(x_1)+f(x_2)\right) + \cdots + \frac{1}{2}h\left(f(x_{n-2}) + f(x_{n-1})\right) + \frac{1}{2}h\left(f(x_{n-1}) + f(x_n)\right)$$

$$\Rightarrow \quad I = \left(\frac{h}{2}\right)\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

where $h = \dfrac{b-a}{n}$, h:- step size

$$E_t = \frac{-h^3}{12} f''(\xi) \quad , \quad a < \xi < b$$

$$E_a = \frac{-(b-a)}{12} h^2 \underbrace{\bar{f}''(x)}$$

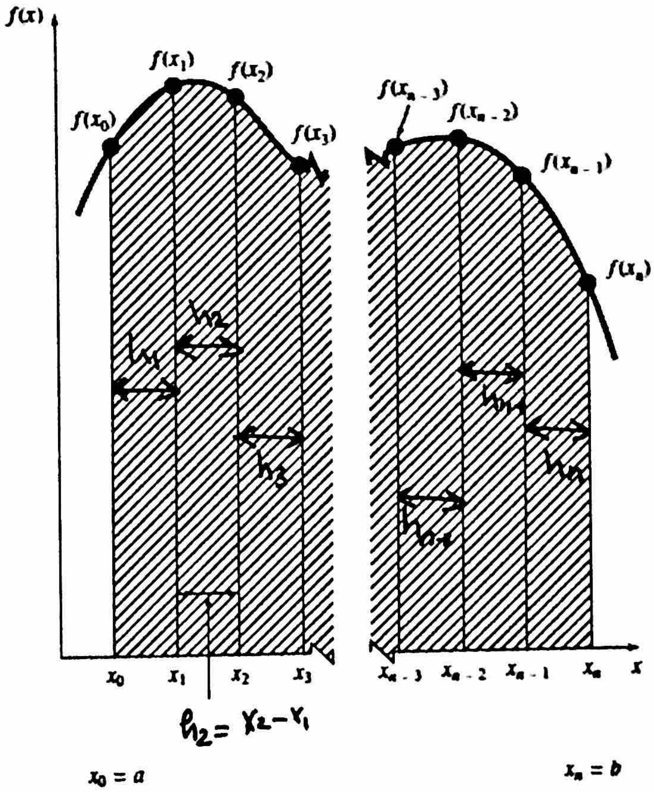
↳ mean of $f''(x)$, as given previously.

* For the case of unequal segments (widths)

$$I = \frac{f(x_1) + f(x_0)}{2} h_1 + \frac{f(x_2) + f(x_1)}{2} h_2$$

$$+ \cdots + \frac{f(x_{n-1}) + f(x_{n-2})}{2} h_{n-1}$$

$$+ \frac{f(x_n) + f(x_{n-1})}{2} h_n$$



Example For $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Use Trapezadal rule to evaluate $\int_0^{0.8} f(x)\,dx$, If

① $n = 1$
② $n = 2$

and compute E_a for both cases.

Solution

① $n=1$

$$I \cong \frac{1}{2}(b-a)\left[f(b) + f(a)\right] \quad , \quad b = 0.8, \; a = 0$$

$$I = \frac{1}{2}(0.8-0)\left[f(0.8) + f(0)\right] \quad , \quad f(0) = 0.2, \; f(0.8) = 0.232$$

$$I = 0.4(0.2 + 0.232) \Rightarrow I = 0.1728$$

$$E_a = \frac{-1}{12}\overline{f''(x)}\,(b-a)^3 \quad , \quad f''(x) = -400 + 4050x - 10800x^2 + 8000x^3$$

$$\overline{f''(x)} = \frac{\int_a^b f''(x)\,dx}{b-a} = \frac{\int_0^{0.8}(-400 + 4050x - 10800x^2 + 8000x^3)\,dx}{0.8 - 0}$$

$$\overline{f''(x)} = -60$$

$$\Rightarrow E_a = \frac{-1}{12}(-60)(0.8-0)^3$$

$$E_a = 2.56$$

② $n=2$

$$I=\left(\frac{h}{2}\right)\left[f(x_0)+2\sum_{i=1}^{n-1}f(x_i)+f(x_n)\right] \quad , \quad h=\frac{b-a}{n}=\frac{b-a}{2}$$

$h = 0.4$

$$=\left(\frac{h}{2}\right)\left[f(x_0)+2f(x_1)+f(x_2)\right] \quad ,$$

$x_0 = 0$ (a)
$x_1 = 0.4 \Rightarrow x_1 = x_0 + h$
$x_2 = 0.8 \Rightarrow x_2 = x_1 + h$
(b) $= x_0 + 2h$

$$=\left(\frac{0.4}{2}\right)\left[f(0)+2f(0.4)+f(0.8)\right]$$

$f(0) = 0.2$
$f(0.4) = 2.456$
$f(0.8) = 0.232$

$$I = 1.0688$$

$$E_a = \frac{-(b-a)}{12}h^2\overline{f''(x)} = \frac{0.8-0}{12}(0.4)^2(-60) = 0.64$$

remember $\quad n=1 \Rightarrow E_a = 2.56$
$\quad n=2 \Rightarrow E_a = 0.64$

For $n=1$ $\quad E_a = \frac{-(b-a)^3}{12}\overline{f''(x)}$

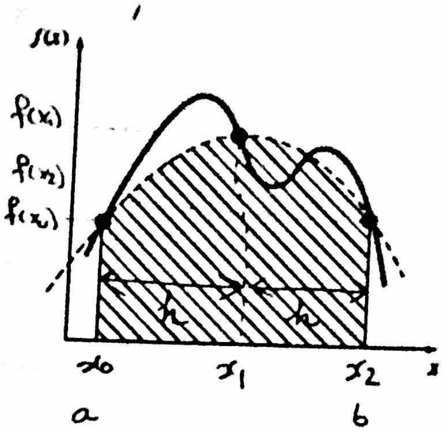
$n=2$ $\quad E_a = \frac{-(b-a)^3}{12n^2}\overline{f''(x)}$

If n doubled $\Rightarrow E_{a_{new}} = \frac{1}{4}E_{a_{old}}$

*Chapter 21 : Numerical Integration

21.2 Simpson's Rules

21.2.1 1/3 Simpson's Rule



$$I = \int_a^b \overset{\text{Lagrange interpolation}}{f_2(x)}\, dx$$

$$= \int_a^b \left[\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right) f(x_0) + \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right) f(x_1) + \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right) f(x_2) \right] dx$$

Chapter 18

$$f_n(x) = \sum_{i=0}^{n} f(x_i)\, L_i(x)$$

$$L_i(x) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$I = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right], \quad h = \frac{b-a}{2}$$

True Error (E_t)

$$E_t = -\frac{1}{90} h^5 \overset{\to \text{4th derivative}}{f''''(\xi)} \qquad a < \xi < b$$

Approximate Error (E_a)

$$E_a = -\frac{1}{90} h^5 \overline{f''''(x)}, \quad \overline{f''''(x)} = \frac{\int_a^b f''''(x)\, dx}{b-a}$$

In 1/3 Simpson's Rule, we always get exact value of the Integration If $f(x)$ is polynomial of degree "3" or Lower

$f''''(x) = 0$

For multisegment 1/3 Simpson's rule

$$I = \frac{h}{3}\left[f(x_0) + 4\sum_{i=odd}^{n-1} f(x_i) + 2\sum_{j=even}^{n-2} f(x_j) + f(x_n)\right]$$

where $h = \frac{b-a}{n}$, $n = \#$ of segments , $E_a = \frac{-(b-a)^5}{180\,n^4}\overline{f''''(x)}$

$= \frac{-(b-a)}{180} h^4 \overline{f''''(x)}$

Example $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Use 1/3 Simpson's rule to evaluate

$\int_0^{0.8} f(x)dx$, for (a) $n=1$ and (b) $n=4$, compute (E_a)

for Both Cases

Solution

(a) $n=1$, $I = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$

$x_0 = a = 0$

$x_1 = x_0 + h = 0.4$

(b) $x_2 = x_1 + h = 0.8$

$x_0 + 2h$

$h = \frac{b-a}{2}$

$= \frac{0.8-0}{2}$

$= 0.4$

$\Rightarrow I = \frac{0.4}{3}\left[f(0) + 4f(0.4) + f(0.8)\right]$

$I = 1.367$

$E_a = -\frac{1}{90} h^5 \overline{f''''(x)}$, $\overline{f''''(x)} = \frac{\int_a^b f''''(x)dx}{b-a} = -2400$

$= -\frac{1}{90}(0.4)^5(-2400) = 0.273$

(b) $n = 4$

$$I = \frac{h}{3}\left[f(x_0) + 4\sum_{i=odd}^{n-1=4} f(x_i) + 2\sum_{j=even}^{n-2=4} f(x_j) + \underset{4}{f(x_n)}\right]$$

$$h = \frac{b-a}{n} = \frac{0.8-0}{4} \Rightarrow h = 0.2$$

$x_0 = a = 0$, $x_1 = x_0 + h = 0.2$, $x_2 = x_1 + h = 0.4$, $x_3 = x_2 + h = 0.6$, $x_4 = b = 0.8$ ($x_4 = x_3 + h$)

$$I = \frac{0.2}{3}\left[f(x_0) + 4\left(f(x_1) + f(x_3)\right) + 2\left(f(x_2)\right) + f(x_n)\right]$$

$$= \frac{0.2}{3}\left[f(0) + 4\left(f(0.2) + f(0.6)\right) + 2f(0.4) + f(0.8)\right]$$

$f(0) = 0.2$, $f(0.2) = 1.288$, $f(0.4) = 2.456$, $f(0.6) = 3.464$, $f(0.8) = 0.232$

$$I = 1.623$$

$$E_a = \frac{-(b-a)^5}{(180)n^4} f''''(x) = -\frac{(0.8-0)^5}{(180)(4)^4}(-2400) = 0.017$$

compare to Trapezoidal Rule, 1/3 is much more accurate (Less Ea values)

21.2.3 3/8 Simpson's Rule

$$I = \int_a^b f_3(x)\,dx$$, we can get $f_3(x)$ from Chapter 18

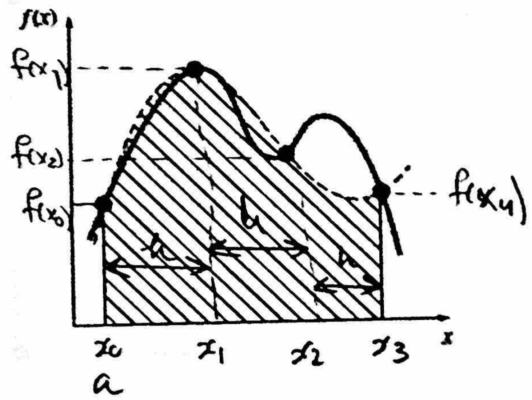
$$I = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$h = \frac{b-a}{3}$$

$x_0 = a$, $x_3 = b$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$



True Error

$$E_t = -\frac{(b-a)^5}{6480} f''''(\xi) \quad , a < \xi < b$$

Approximate Error

$$E_a = -\frac{(b-a)^5}{6480} \overline{f''''(x)}$$

3/8 Simpson's rule gives exact value if $f(x)$ polynomial of degree 3 or less.

EXAMPLE 21.6 Simpson's 3/8 Rule

Problem Statement.

Use Simpson's 3/8 rule to integrate

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

from $a = 0$ to $b = 0.8$.

Solution.

A single application of Simpson's 3/8 rule requires four equally spaced points:

$f(0) = 0.2 \qquad f(0.2667) = 1.432724$

$f(0.5333) = 3.487177 \qquad f(0.8) = 0.232$

Using Eq. (21.20),

$$I \cong 0.8\frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8} = 1.519170$$

$$E_a = -\frac{(0.8)^5}{6480}(-2400) = 0.1213630$$