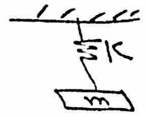
* Review for chapter one

* Chapter one

→ Free vibration (No Forces)

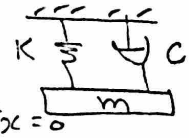
→ Undamped

$$m\ddot{x} + Kx = 0$$
$$\ddot{x} + \omega_n^2 x = 0$$



→ Damped

$$m\ddot{x} + c\dot{x} + Kx = 0$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$



Homogeneous ODE

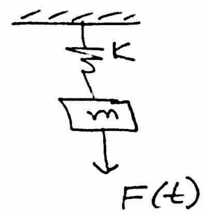
$$x(t) = C e^{\lambda t}$$

* Chapter Two

→ Forced Vibration (with Forces)

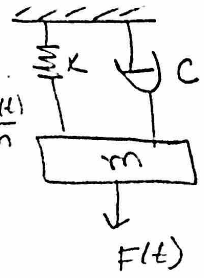
→ Undamped

$$m\ddot{x} + Kx = F(t)$$
$$\ddot{x} + \omega_n^2 x = \frac{F(t)}{m}$$



→ Damped

$$m\ddot{x} + c\dot{x} + Kx = F(t)$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$



Non-Homogeneous ODE

$$x(t) = x_h(t) + x_p(t)$$

→ particular

Homog.

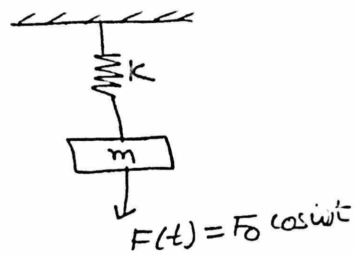
→ Same as chapter 1

* Forced Vibration of undamped System

(Harmonic Forces Excitation)

F_0 = Force amplitude (N)

ω = Excitation Frequency (rad/s)



Equation of motion

$$m\ddot{x} + Kx = F_0 \cos\omega t$$

$$\ddot{x} + \omega_n^2 x = f_0 \cos\omega t \quad , \quad f_0 = \frac{F_0}{m}$$

Solution

$$x(t) = \underbrace{x_h(t)}_{\text{Homog.}} + \underbrace{x_p(t)}_{\text{particular}}$$

we know, (From ch.1) $\longrightarrow \ddot{x} + \omega_n^2 x = 0$

$$x_h(t) = A\cos\omega_n t + B\sin\omega_n t$$

For $x_p(t)$

$$x_p(t) = \alpha\cos\omega t + \beta\sin\omega t$$

Substitute in EOM

$$x_p(t) = \alpha\cos\omega t + \beta\sin\omega t$$

$$\dot{x}_p(t) = -\omega\alpha\sin\omega t + \omega\beta\cos\omega t$$

$$\ddot{x}_p(t) = -\omega^2\alpha\cos\omega t - \omega^2\beta\sin\omega t$$

$$-\omega^2\alpha\cos\omega t - \omega^2\beta\sin\omega t + \omega_n^2\alpha\cos\omega t + \omega_n^2\beta\sin\omega t = f_0\cos\omega t$$

terms with $\cos\omega t$

$$-\omega^2\alpha\cancel{\cos\omega t} + \omega_n^2\alpha\cancel{\cos\omega t} = f_0\cancel{\cos\omega t}$$

$$\Rightarrow \boxed{\alpha = \frac{f_0}{\omega_n^2 - \omega^2}}$$

terms with, $\sin\omega t$

$$-\omega^2\beta\cancel{\sin\omega t} + \omega_n^2\beta\cancel{\sin\omega t} = 0$$

$$\Rightarrow \boxed{\beta = 0}$$

$$\Rightarrow x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

Total solution

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

A and B constants from Initial conditions.

For initial conditions

$x(0) = x_0$ and $\dot{x}(0) = v_0 \Rightarrow$ Find A and B

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cdot \cos\omega t$$

$$x(0) = x_0 = A + 0 + \frac{f_0}{\omega_n^2 - \omega^2}$$

$$\Rightarrow A = x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \quad , \quad f_0 = F_0/m$$

$\dot{x}(0) = v_0 \Rightarrow$

$$\dot{x}(t) = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t - \omega\frac{f_0}{\omega_n^2 - \omega^2}\sin\omega t$$

$$\dot{x}(0) = v_0 = 0 + \omega_n B - 0$$

$$\Rightarrow B = \frac{v_0}{\omega_n}$$

$$\Rightarrow x(t) = \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right)\cos\omega_n t + \frac{v_0}{\omega_n} \cdot \sin\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

$$\omega_n = \sqrt{\frac{k}{m}} \quad , \quad f_0 = F_0/m$$

* <u>If</u> $\omega_n = \omega$, Natural Freq = Excitation Freq.

$$x(t) \longrightarrow \infty$$

(Resonance)

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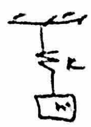
* <u>If</u> $F(t) = F_0\cos\omega t = 0 \rightarrow F_0 = 0$

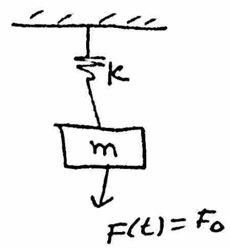
$$x(t) = x_0\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

Back to free-undamped system



* Constant Force, $F(t) = F_0$, $\omega = 0$

$x(0) = x_0$
$\dot{x}(0) = v_0$



Equation of motion

$$\ddot{x} + \omega_n x = f_0 \quad , \quad f_0 = \frac{F_0}{m}$$

$$x(t) = \left(x_0 - \frac{f_0}{\omega_n^2}\right)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t + \frac{f_0}{\omega_n^2}$$

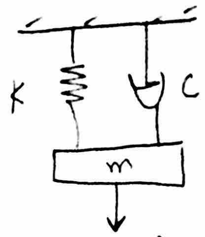
Another Form: Find $x(t)$, If $F(t) = F_0 \sin\omega t$

to practice

$F(t) = F_0 \sin\omega t$

practice

2.2 Harmonic Excitation of Damped Systems



$F(t) = F_0 \cos \omega t$

Amp

Excitation Freq

Equation of motion

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 \cos \omega t$$

$$2\zeta\omega_n = \frac{c}{m} \quad , \quad \omega_n^2 = \frac{k}{m} \quad , \quad f_0 = \frac{F_0}{m}$$

Solution

$$x(t) = \underbrace{x_h(t)}_{\text{Homog.}} + \underbrace{x_p(t)}_{\text{particular}}$$

$$x_p(t) = \alpha \cos \omega t + \beta \sin \omega t$$

$$\dot{x}_p(t) = -\omega\alpha \sin \omega t + \omega\beta \cos \omega t$$

$$\ddot{x}_p(t) = -\omega^2 \alpha \cos \omega t - \omega^2 \beta \sin \omega t$$

Substitute in governing equation

$$-\omega^2\alpha \cos \omega t - \omega^2 \beta \sin \omega t + 2\zeta\omega_n(-\omega\alpha \sin \omega t + \omega\beta \cos \omega t)$$

$$+ \omega_n^2 (\alpha \cos \omega t + \beta \sin \omega t) = f_0 \cos \omega t$$

Collect

$$(-\omega^2\alpha + 2\zeta\omega\omega_n\beta + \omega_n^2\alpha)\cancel{\cos \omega t} = f_0 \cancel{\cos \omega t}$$

$$(-\omega^2\beta - 2\zeta\omega\omega_n\alpha + \omega_n^2\beta)\cancel{\sin \omega t} = 0$$

$$(\omega_n^2 - \omega^2)\alpha + 2\zeta\omega\omega_n\beta = f_0$$

$$-2\zeta\omega\omega_n\alpha + (\omega_n^2 - \omega^2)\beta = 0$$

$$\begin{bmatrix} \omega_n^2 - \omega^2 & 2\zeta\omega\omega_n \\ -2\zeta\omega\omega_n & \omega_n^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} f_0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad \alpha = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} \qquad x_p(t) = \alpha \cos \omega t + \beta \sin \omega t$$

$$\beta = \frac{2\zeta\omega\omega_n f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

Cont'd

We can write the $x_p(t)$, as:

$$x_p(t) = X\cos(\omega t - \theta)$$

where $X = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}} \quad , \quad \theta = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$

Then total Solution

$$x(t) = x_h(t) + x_p(t)$$

$\zeta < 1$

$\zeta = 1$

$\zeta > 1$

, remember $\zeta > 0$, If $\zeta = 0$ (undamped system)

* $x_h(t)$ is called transient Solution, $x_h \to 0$ when $t \to \infty$
* $x_p(t)$ is called steady-state solution, $x_p \neq 0$ for any t

Because $x_h \to 0$ when $t \to \infty$, we focus on the steady state response (Solution).

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}} \quad , \quad \theta = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$

remember $f_0 = F_0/m$ and take ω_n^2 outside $\left(* \frac{\omega_n^2}{f_0}\right)$

and $\theta \Rightarrow \frac{1/\omega_n^2}{1/\omega_n^2}$

$$\Rightarrow \frac{X\omega_n^2}{f_0} = \frac{XK}{F_0} = \frac{1}{\sqrt{(1-r^2) + (2\zeta r)^2}}$$

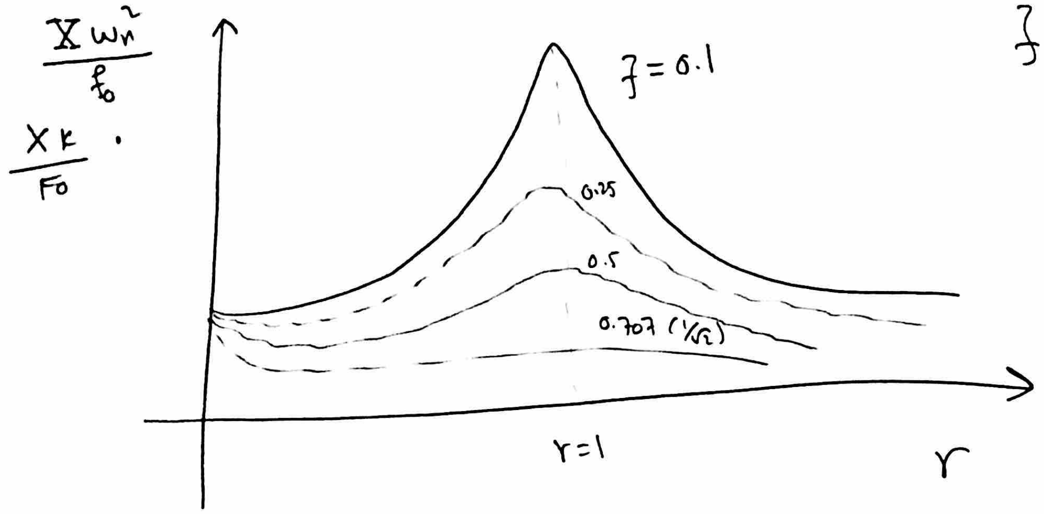
(Normalized force magnitude)

$$\Rightarrow \theta = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \quad , \quad r = \frac{\omega}{\omega_n}$$

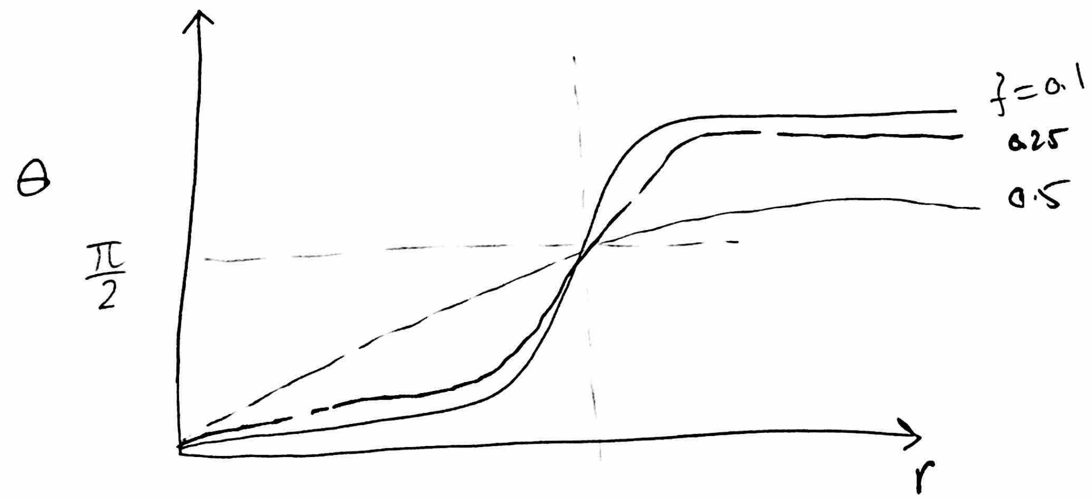
Freq ratio

(3)

If $\omega = \omega_n$ or $r = 1$ $\Rightarrow$ resonance



$\zeta \downarrow$ peak sharpness $\uparrow$



$\zeta \uparrow$ linear $\uparrow$

Example Find r that makes $\frac{XK}{F_0}$ maximum

Solution

$$\frac{XK}{F_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\frac{d}{dr}\left(\frac{XK}{F_0}\right) = \frac{d}{dr}\left(\left[(1-r^2)^2 + (2\zeta r)^2\right]^{-1/2}\right) = 0$$

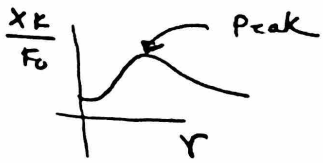
$$\Rightarrow \quad r_{peak} = \sqrt{1-2\zeta^2} \qquad \text{but we know} \quad r_{peak} = \frac{\omega}{\omega_n} = 1$$

So, $\quad r = \sqrt{1-2\zeta^2} = \frac{\omega}{\omega_n}$

$$\Rightarrow \zeta < \frac{1}{\sqrt{2}}$$

What does that mean?

$\rightarrow$ If $\zeta < \frac{1}{\sqrt{2}}$ we will have peak



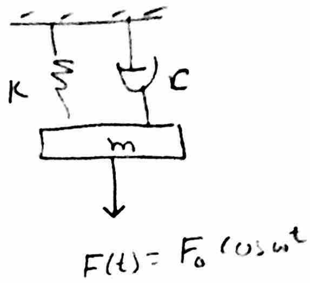
$\zeta \geq \frac{1}{\sqrt{2}}$ No peak

See plots page 139 and 141

Example

$m = 49.2 \times 10^{-3}$ kg, $c = 0.11$ kg/s, $k = 8578$ N/m

$\omega = 132$ rad/s, $f_o = \frac{F_o}{m} = 10$ N/kg



① Find the steady-state response amplitude

Solution

$x_{ss}(t) = X \cos(\omega t - \theta)$

$$X = \frac{f_o}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}}$$

$\Rightarrow \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8578}{49.2 \times 10^{-3}}} = 132$ rad/s

$\zeta = \frac{c}{2m\omega_n} = \frac{0.11}{(2)(49.2)(10^{-3})(132)} = 0.0085$

$$= \frac{10}{\sqrt{(132^2 - 132^2)^2 + (2(0.0085)(132)(132))^2}} \Rightarrow X = 0.034 \text{ m}$$

② If $\omega_n = 125$ rad/s, Find X

$$X = \frac{10}{\sqrt{(125^2 - 132^2)^2 + (2(0.0085)(132)(125))^2}} = 0.005 \text{ mm}$$

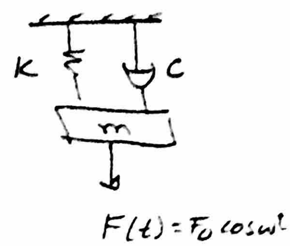
See if $\omega_n = 132$ (resonance) $\quad X = 0.034$ m

If $\omega_n = 125$ (not reson.) $\quad X = 0.005$ m

$\left.\begin{matrix} X = 0.034 \text{ m} \\ X = 0.005 \text{ m} \end{matrix}\right\}$ seven times

diff 5 rad/s $\rightarrow$ Caused much lower X

* Underdamped system



$$m\ddot{x} + c\dot{x} + Kx = F_0 \cos\omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos\omega t, \quad f_0 = F_0/m$$

Solution

$$x(t) = x_h(t) + x_p(t)$$

If $\zeta < 1$ (underdamped)

$$x_h(t) = e^{-\zeta\omega_n t}(A\cos\omega_n t + B\sin\omega_n t)$$

$$x_p(t) = \alpha\cos\omega t + \beta\sin\omega t$$

$$\alpha = \frac{(\omega_n^2-\omega^2)f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}$$

$$\beta = \frac{2\zeta\omega\omega_n f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}$$

total solution

$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_n t + B\sin\omega_n t) + \alpha\cos\omega t + \beta\sin\omega t$$

A and B constants from Initial conditions.

$x(0) = x_0$, $\dot{x}(0) = v_0$ $\Rightarrow$ Find A and B

$$A = x_0 - \frac{f_0(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}$$

$$B = \frac{\zeta\omega_n}{\omega_d}\left(x_0 - \frac{f_0(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\right) - \left(\frac{2\zeta\omega^2\omega_n}{(\omega_n^2-\omega^2)^2+(2\zeta\omega\omega_n)^2}\right)\cdot\frac{1}{\omega_d} + \frac{v_0}{\omega_d}$$

$$B = \frac{\zeta\omega_n}{\omega_d}A + \frac{v_0}{\omega_d} - \frac{\omega\beta}{\omega_d}$$

Ask them to do the same for $\zeta = 1$, $\zeta > 1$

Combine homogeneous and particular solutions to obtain total solution $x(0) = x_0$, $\dot{x}(0) = v_0$