* Principal values and principal Directions (or), Eigenvalues and Eigen vectors

For a matrix $[A]$, we can find the principal values (λ) and the associated principal direction $\{n\}$, such that

$$[A]\{n\} = \lambda\{n\} \quad , \quad [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

To find λ and $\{n\}$

$$\Rightarrow [A]\{n\} - \lambda\{n\} = 0$$

$$\Rightarrow \underbrace{([A] - \lambda[I])}_{det = 0}\underbrace{\{n\}}_{\neq 0} = 0$$

$$[A] - \lambda[I] = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$$det([A] - \lambda[I]) = 0 \Rightarrow \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

characteristic equation

I_1, I_2 and I_3 invariants $\qquad \Rightarrow \lambda_1, \lambda_2$ and λ_3 (Eigenvalues)

$$I_1 = a_{11} + a_{22} + a_{33} = tr(A),$$

$$I_2 = \frac{1}{2}\left[tr^2(A) + tr(A^2) \right], \quad A^2 = [A][A]$$

$$I_3 = det(A)$$

* The Solution of characteristic Equation gives

λ_1, λ_2 and λ_3 $\Rightarrow$ Eigenvalues (principal values)

such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, all λ_i $(i=1,2,3)$ are real

* Cases of λ_i

① $\lambda_1 \neq \lambda_2 \neq \lambda_3 \Rightarrow$ In this case, all principal directions (eignvectors) are unique and not related to each other.

② If two values are equal $(\lambda_1 \neq \lambda_2 = \lambda_3)$. In this case, the principal direction $\{n^{(1)}\}$ of λ_1 is unique and the principal directions of λ_2 and λ_3 ($\{n^{(2)}\}$ and $\{n^{(3)}\}$ are related by a Scalar "C".

$$\{n^{(2)}\} = c\{n^{(3)}\}$$

③ All principal values are equal, All eignvectors are related

$$\{n^{(1)}\} = A\{n^{(2)}\} = B\{n^{(3)}\}$$

<u>Example</u>: Find the principal values and principal directions

$$[A] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

<u>Solution</u>

① principal values

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

$$I_1 = tr(A) = 2 + 3 + -3 = 2 \qquad \boxed{I_1 = 2}$$

$$I_2 = \frac{1}{2}\left(tr^2(A) - tr(A^2)\right)$$

$$= \frac{1}{2}\left((2)^2 - 54\right) = -25$$

$$[A]^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

$$I_3 = det(A) = -50$$

$$\Rightarrow \lambda_3 - 2\lambda^2 - 25\lambda + 50 = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = 2, \quad \lambda_3 = -5$$

* Now, we find principal directions

$\lambda_1 = 5$

$\Rightarrow [A]\{n^{(1)}\} = \lambda_1 \{n^{(1)}\}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix} \begin{Bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{Bmatrix} = 5 \begin{Bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{Bmatrix}$$

① $2 n_1^{(1)} = 5 n_1^{(1)} \Rightarrow 3 n_1^{(1)} = 0 \Rightarrow \boxed{n_1^{(1)} = 0}$ ①

② $3 n_2^{(1)} + 4 n_3^{(1)} = 5 n_2^{(1)} \Rightarrow -2 n_2^{(1)} + 4 n_3^{(1)} = 0$

$\boxed{-n_2^{(1)} + 2 n_3^{(1)} = 0}$ ②

③ $4 n_2^{(1)} - 3 n_3^{(1)} = 5 n_3^{(1)} \Rightarrow 4 n_2^{(1)} - 8 n_3^{(2)} = 0$

$\boxed{-n_2^{(1)} + 2 n_3^{(1)} = 0}$ ③

Eq(2) and Eq(3) are the same $\Rightarrow n_2^{(1)} = 2 n_3^{(1)}$

let $n_3^{(1)} = 1 \Rightarrow \boxed{n_2^{(1)} = 2}$

$$\Rightarrow \{n^{(1)}\} = \begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix} \Rightarrow \overrightarrow{n^{(1)}} = 0\hat{i} + 2\hat{j} + \hat{k}$$

$$\overrightarrow{n^{(1)}} = 2\hat{j} + \hat{k}$$

First principal directi
(Eigen value)

Length of $\overrightarrow{n^{(1)}} = \sqrt{(0)^2 + (2)^2 + (1)^2}$

$= \sqrt{5}$

If we want to normalize $\overrightarrow{n^{(1)}}$, or find $\overrightarrow{N^{(1)}}$

$$\{N^{(1)}\} = \begin{Bmatrix} N_1^{(1)} \\ N_2^{(1)} \\ N_3^{(1)} \end{Bmatrix}, \text{ where } N_i^{(1)} = \frac{n_i^{(1)}}{\sqrt{(n_1^{(1)})^2 + (n_2^{(1)})^2 + (n_3^{(1)})^2}}$$

Normalized 1st eigenvector

$\Rightarrow$ length $= 1$

length

So, $\sqrt{(0)^2 + (2)^2 + (1)^2} = \sqrt{5}$

$$\Rightarrow \{N^{(1)}\} = \begin{Bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{Bmatrix}$$

To find $\{n^{(2)}\}$,

$\lambda_2 = 0 \quad \Rightarrow \{n^{(2)}\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ using same procedure

Normalized vector $\{N^{(2)}\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$

For $\{n^{(3)}\}$, $\lambda_3 = -5$

$$\Rightarrow \{n^{(3)}\} = \begin{Bmatrix} 0 \\ 1 \\ -2 \end{Bmatrix} \Rightarrow \{N^{(3)}\} = \begin{Bmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{Bmatrix}$$

Note

$$[Q^n] = \left[\{N^{(1)}\}^T ; \{N^{(2)}\}^T ; \{N^{(3)}\}^T \right]$$ ⇒ All principal directions entered row-by-row

transformation matrix

$$[A'] = [Q^n][A][Q^n]^T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

For $[A']$,

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$I_3 = (\lambda_1)(\lambda_2)(\lambda_3)$$

In this example, $$[Q^n] = \begin{bmatrix} 0 & 2/\sqrt{5} & -1/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & -2/\sqrt{5} \end{bmatrix}$$

$$\Rightarrow [A'] = [Q^n][A][Q^n]^T = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

to check your Solution