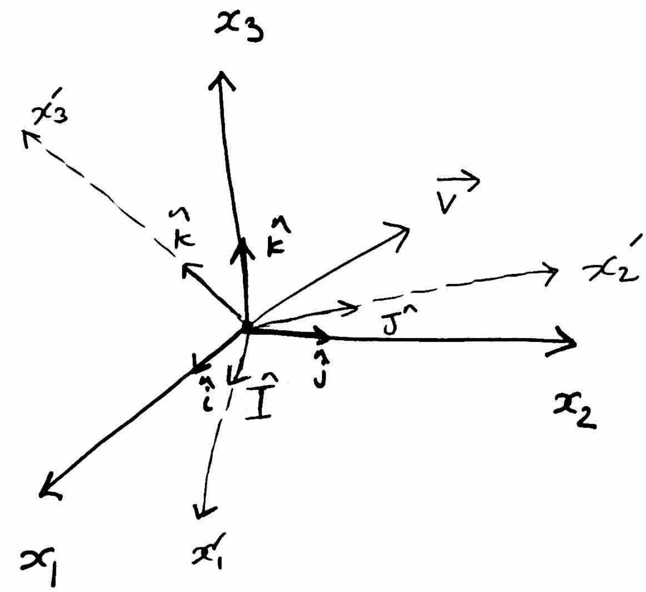
* Coordinate transformation

axis $\Rightarrow x_1, x_2, x_3$

Unit vectors $\hat{i}, \hat{j}, \hat{k}$



Vector $\vec{V}$

$$\vec{V} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

* For a new coordinate system x_1', x_2', x_3'

Unit vectors $\hat{I}, \hat{J}, \hat{K}$

* We need to transform $\vec{V}$ to the new coordinate system or write it in terms of $\hat{I}, \hat{J}, \hat{K}$

$$\vec{V} = v_x' \hat{I} + v_y' \hat{J} + v_z' \hat{K}$$

$$\{V'\} = [Q]\{V\}$$

transformation matrix, or

direction cosines matrix

$[Q] = Q_{ij} = \cos(x_i', x_j) \rightarrow$ cosines between new axies and old axes

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) & \cos(x_1', x_3) \\ \cos(x_2', x_1) & \cos(x_2', x_2) & \cos(x_2', x_3) \\ \cos(x_3', x_1) & \cos(x_3', x_2) & \cos(x_3', x_3) \end{bmatrix}$$

For a matrix $a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$[a'] = [Q][a][Q]^T$$

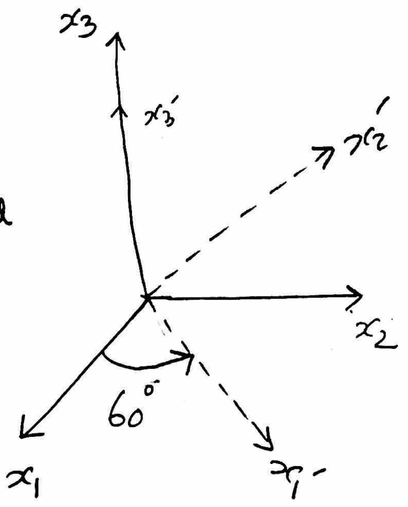
Properties of [Q]

$$[Q]^{-1} = [Q]^T$$

$$\Rightarrow [Q][Q]^{-1} = [Q][Q]^T = [I]$$

Example

For a rotation of 60° counter-clock-wise about x_3 axis. Find the transformed state of



$$\{v\} = \begin{Bmatrix} 1 \\ 4 \\ 2 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

Solution

$$[Q] = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) & \cos(x_1', x_3) \\ \cos(x_2', x_1) & \cos(x_2', x_2) & \cos(x_2', x_3) \\ \cos(x_3', x_1) & \cos(x_3', x_2) & \cos(x_3', x_3) \end{bmatrix}$$

$$= \begin{bmatrix} \cos 60^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 150^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1 $\{v'\} = [Q]\{v\}$

$$= \begin{bmatrix} 1/2 & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 4 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 3.9641 \\ 1.1340 \\ 2 \end{Bmatrix}$$

$$[A'] = [Q][A][Q]^T$$

$$= \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\frac{\sqrt{3}}{2} & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7/4 & \frac{\sqrt{3}}{4} & 3/2 + \sqrt{3} \\ \sqrt{3}/4 & 5/4 & 1 - 3\sqrt{3}/2 \\ \frac{3}{2} & 1 - 3\sqrt{3}/2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1.75 & 0.433 & 3.232 \\ 0.433 & 1.25 & -1.598 \\ 3.232 & -1.598 & 4 \end{bmatrix}$$