Chapter 10 LU Decomposition

For a system of linear Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad , \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$[A]\{x\} = \{B\}$$

Using LU-decomposing

$[A] \rightarrow$ Decompose $\rightarrow$ $[L]$, Lower triangular matrix

$[A] \rightarrow$ Decompose $\rightarrow$ $[U]$, Upper triangular matrix

$$[L][U] = [A]$$

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

, we get $[U]$ from row operations of Gauss Elimination

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix},$$

we need to find: l_{21}, l_{31} and l_{32}

- How?

- Remember, $[L][U] = [A]$

$$[L][U] = [A]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- $\underline{\text{row } 2 \times \text{ column } 1} = a_{21}$

$$a_{21} = L_{21} a_{11} + (1)(0) + (0)(0) \Rightarrow \boxed{L_{21} = \frac{a_{21}}{a_{11}}}$$

- $\underline{\text{row } 3 \times \text{ Column } 1} = a_{31}$

$$a_{31} = L_{31} a_{11} + L_{32}(0) + (1)(0) \Rightarrow \boxed{L_{31} = \frac{a_{31}}{a_{11}}}$$

- $\underline{\text{row } 3 \times \text{ column } 2} = a_{32}$

$$a_{32} = L_{31} a_{12} + L_{32} a_{22}' + (1)(0) \Rightarrow \boxed{L_{32} = \frac{a_{32} - L_{31} a_{12}}{a_{22}'}}$$

$$[L]\underbrace{[U]\{x\}}_{\{D\}} = \{B\}$$

$\{D\}$ – Intermediate vector , $\{D\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$

$$[U]\{x\} = \{D\}$$

$[L]\{D\} = \{B\} \rightarrow$ get vector $\{D\}$

Then Back to $\{D\} = [U]\{x\} \Rightarrow$ get vector $\{x\}$ → Solution

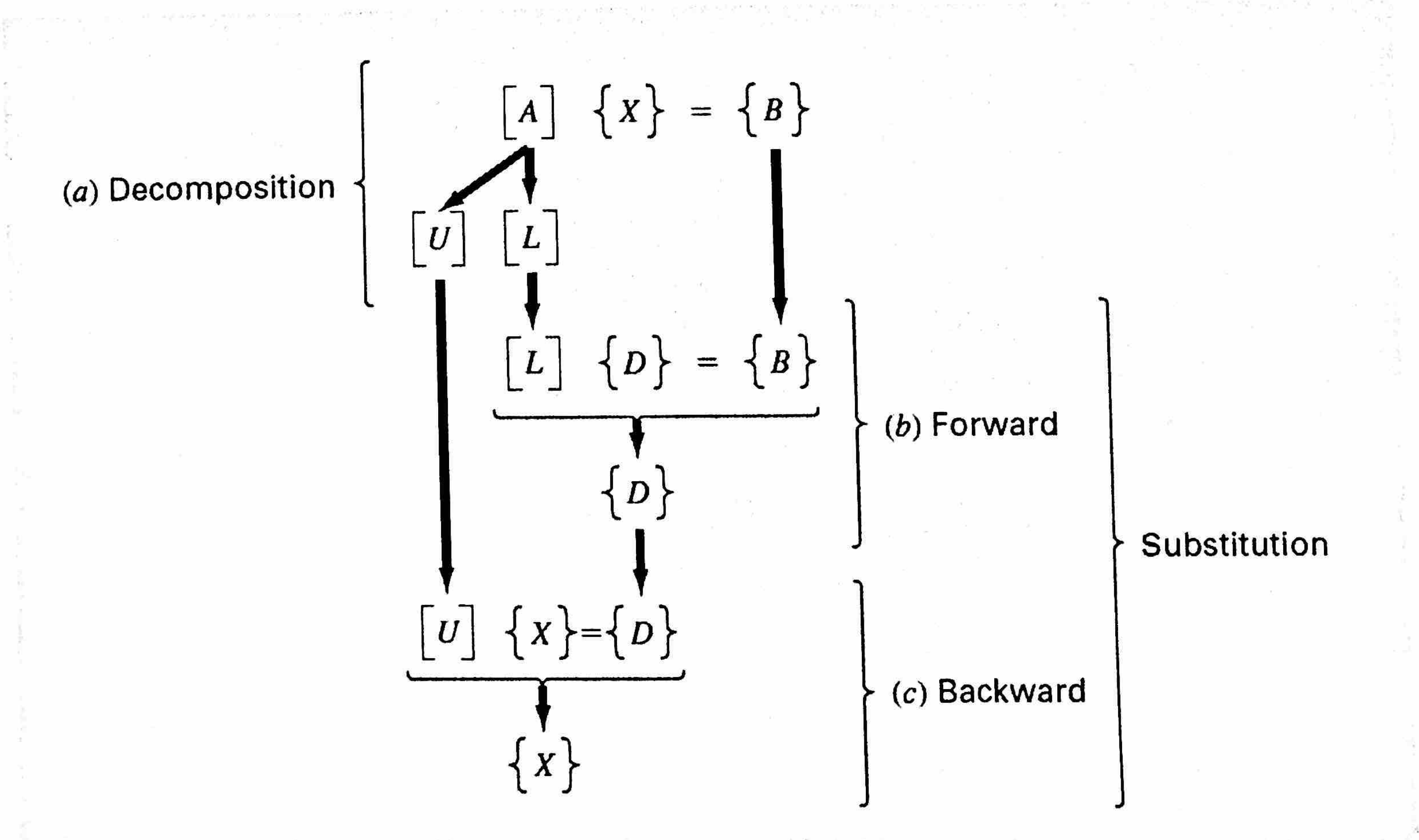
$$[A]\{x\}=\{B\}$$

$$\rightarrow [L][U]=[A]$$

$$\rightarrow [L]\underbrace{[U]\{x\}}_{\{D\}}=\{B\}$$

$$\rightarrow [L]\{D\}=\{B\} \rightarrow \text{get } \{D\}$$

$$\rightarrow [U]\{x\}=\{D\} \rightarrow \text{get } \{x\}$$



Example: Use LU decomposition to solve

$$x_1 + x_2 + x_3 = 6$$
$$2x_1 + 3x_2 + 4x_3 = 20$$
$$3x_1 + 4x_2 + 2x_3 = 17$$

Solution

Matrix Form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 20 \\ 17 \end{Bmatrix}$$

$$[A] \quad \{X\} \quad = \quad \{B\}$$

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{matrix} \nearrow [L] \\ \searrow [U] \end{matrix}$$

$$[U] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix},$$

$$L_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$$

$$L_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{1} = 3$$

$$L_{32} = \frac{a_{32} - L_{31}\, a_{12}}{a_{22}'} = \frac{4 - (3)(1)}{1} = 1$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

$$[L]\{d\} = \{B\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 20 \\ 17 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 20 \\ 17 \end{Bmatrix}$$

$$\boxed{d_1 = 6}$$

$$2d_1 + d_2 = 20 \Rightarrow d_2 = 20 - 2d_1 = 20 - (2)(6)$$

$$\boxed{d_2 = 8}$$

$$3d_1 + d_2 + d_3 = 17 \Rightarrow d_3 = 17 - 3d_1 - d_2 = 17 - (3)(6) - (8)$$

$$\boxed{d_3 = -9}$$

$$[U]\{x\} = \{D\}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 8 \\ -9 \end{Bmatrix}$$

Backward substitution

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

Chapter 10 LU method

$$[A] \rightarrow [L], \quad [A] \rightarrow [U] \qquad [L][U] = [A]$$

10.2 Matrix Inverse

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad [A]^{-1} = \begin{bmatrix} a_{11}^* & a_{12}^* & a_{13}^* \\ a_{21}^* & a_{22}^* & a_{23}^* \\ a_{31}^* & a_{32}^* & a_{33}^* \end{bmatrix}$$

$$\begin{array}{ccc} \text{col.1} & \text{col.2} & \text{col.3} \end{array}$$

* We can use LU to obtain matrix inverse

* In LU, we find inverse, column-by-column

Column 1

$$\{A_1^*\} = \begin{Bmatrix} a_{11}^* \\ a_{21}^* \\ a_{31}^* \end{Bmatrix}$$

$$[L]\{D_1\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \text{Get } \{D_1\}, \quad \{D_1\} = \begin{Bmatrix} d_1^{(1)} \\ d_2^{(1)} \\ d_3^{(1)} \end{Bmatrix}$$

$$[U]\{A_1^*\} = \{D_1\} \Rightarrow \{A_1^*\}$$

Column 2

$$\{A_2^*\} = \begin{Bmatrix} a_{12}^* \\ a_{22}^* \\ a_{32}^* \end{Bmatrix}$$

$$[L]\{D_2\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \Rightarrow \text{Get } \{D_2\}, \quad \{D_2\} = \begin{Bmatrix} d_1^{(2)} \\ d_2^{(2)} \\ d_3^{(2)} \end{Bmatrix}$$

$$[U]\{A_2^*\} = \{D_2\} \Rightarrow \text{Get } \{A_2^*\}$$

Column 3

$$\{A_3^*\} = \begin{Bmatrix} a_{13}^* \\ a_{23}^* \\ a_{33}^* \end{Bmatrix}$$

$$[L]\{D_3\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow \text{Get } \{D_3\}, \quad \{D_3\} = \begin{Bmatrix} d_1^{(3)} \\ d_2^{(3)} \\ d_3^{(3)} \end{Bmatrix}$$

$$[U]\{A_3^*\} = \{D_3\} \Rightarrow \text{Get } \{A_3^*\}$$

$$\Rightarrow [A]^{-1} = \left[\{A_1^*\} \ \{A_2^*\} \ \{A_3^*\} \right]$$

Example

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}, \quad \text{Use LU to find } [A]^{-1}$$

Solution

- we need to find $[A]^{-1}$ column-by-column
- Remember $[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$, $[U] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

- Column 1 $\quad \{A_1^*\} = \begin{Bmatrix} a_{11}^* \\ a_{21}^* \\ a_{31}^* \end{Bmatrix}$

Forward subst.

$$[L]\{D_1\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{Bmatrix} d_1^{(1)} \\ d_2^{(1)} \\ d_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\boxed{d_1^{(1)} = 1}$$

$$2d_1^{(1)} + d_2^{(1)} = 0 \Rightarrow d_2^{(1)} = -2d_1^{(1)} \Rightarrow \boxed{d_2^{(1)} = -2}$$

$$3d_1^{(1)} + d_2^{(1)} + d_3^{(1)} = 0 \Rightarrow d_3^{(1)} = -3d_1^{(1)} - d_2^{(1)} \Rightarrow \boxed{d_3^{(1)} = -1}$$

$$\{D_1\} = \begin{Bmatrix} 1 \\ -2 \\ -1 \end{Bmatrix}$$

Backward Subst.

$$[U]\{A_1^*\} = \{D_1\} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} a_{11}^* \\ a_{21}^* \\ a_{31}^* \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ -1 \end{Bmatrix}$$

$$\boxed{a_{31}^* = 1/3}$$

$$a_{21}^* + 2a_{31}^* = -2 \Rightarrow \boxed{a_{21}^* = -8/3}$$

$$a_{11}^* + a_{21}^* + a_{31}^* = 1 \Rightarrow \boxed{a_{11}^* = 10/3}$$

$$\Rightarrow \{A_1^*\} = \frac{1}{3}\begin{Bmatrix} 10 \\ -8 \\ 1 \end{Bmatrix}$$

* <u>Column 2</u> $\qquad \{A_2^*\} = \begin{Bmatrix} a_{12}^* \\ a_{22}^* \\ a_{32}^* \end{Bmatrix}$

$$[L]\{D_2\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{Bmatrix} d_1^{(2)} \\ d_2^{(2)} \\ d_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\boxed{d_1^{(2)} = 0}$$

$$2d_1^{(2)} + d_2^{(0)} = 1 \Rightarrow \boxed{d_2^{(2)} = 1} \qquad \{D_2\} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$$

$$3d_1^{(2)} + d_2^{(2)} + d_3^{(2)} = 0 \Rightarrow \boxed{d_3^{(2)} = -1}$$

$$[U]\{A_2^*\} = \{D_2\} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} a_{12}^* \\ a_{22}^* \\ a_{32}^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}$$

$$\boxed{a_{32} = 1/3}$$

$$a_{22}^* + 2a_{32}^* = 1 \Rightarrow \boxed{a_{22}^* = 1/3} \qquad \{A_2^*\} = \frac{1}{3}\begin{Bmatrix} -2 \\ 1 \\ 1 \end{Bmatrix}$$

$$a_{12}^* + a_{22}^* + a_{32}^* = 0 \Rightarrow \boxed{a_{12}^* = -2/3}$$

* <u>Column 3</u> $\qquad \{A_3^*\} = \begin{Bmatrix} a_{13}^* \\ a_{23}^* \\ a_{33}^* \end{Bmatrix}$

$$[L]\{D_3\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{Bmatrix} d_1^{(3)} \\ d_2^{(3)} \\ d_3^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$\boxed{d_1^{(3)} = 0}$$

$$2d_1^{(3)} + d_2^{(3)} = 0 \Rightarrow \boxed{d_2^{(3)} = 0} \qquad \{D_3\} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$3d_1^{(3)} + d_2^{(3)} + d_3^{(3)} = 1 \Rightarrow \boxed{d_3^{(3)} = 1}$$

$$[U]\{A_3^*\} = \{D_3\} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} a_{13}^* \\ a_{23}^* \\ a_{33}^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$\boxed{a_{33}^* = -1/3}$$

$$a_{23}^* + 2a_{33}^* = 0 \Rightarrow \boxed{a_{23}^* = 2/3} \qquad \{A_3^*\} = \frac{1}{3}\begin{Bmatrix} -1 \\ 2 \\ -1 \end{Bmatrix}$$

$$a_{13}^* + a_{23}^* + a_{33}^* = 0 \Rightarrow \boxed{a_{13}^* = -1/3}$$

Finally,

$$[A]^{-1} = \left[\{A_1^*\} \; \{A_2^*\} \; \{A_3^*\} \right]$$

$$\Rightarrow \quad [A]^{-1} = \frac{1}{3} \begin{bmatrix} 10 & -2 & -1 \\ -8 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

End of Chapter 10