Chapter 9 : Linear Algebraic Equations : Gauss Elimination

* In this Chapter we will learn how solve small number of linear Equations using.

    1- Cramer's rule

    2- Gauss Elimination

    3- Gauss-Jordan Elimination

* For a system of linear Equation

$$a_{11} x_1 + a_{12} x_2 = c_1$$

$$a_{21} x_1 + a_{22} x_2 = c_2$$

* Can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

$$\underset{2\times2}{[A]} \underset{2\times1}{\{x\}} = \underset{2\times1}{\{c\}}$$ $\Rightarrow$ we need to find $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

* system of three equations:

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = c_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = c_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = c_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

$$\underset{3\times3}{[A]} \underset{3\times1}{\{x\}} = \underset{3\times1}{\{c\}}$$ $\Rightarrow$ we need to find $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$

* Cramer's Rule

- For a system of two equations

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

the variables x_1 and x_2 can be calculated as:

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad , \quad x_2 = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$x_1 = \frac{c_1 a_{22} - c_2 a_{12}}{a_{11} a_{22} - a_{21} a_{12}} \quad , \quad x_2 = \frac{a_{11} c_2 - a_{21} c_1}{a_{11} a_{22} - a_{21} a_{12}}$$

- For 3-equation system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}$$

x_1, x_2 and x_3 can be obtained, as:

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}}{D(A)}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}}{D(A)} \quad , \quad D(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}}{D(A)}$$

* **Example** Use Cramer's rule to solve:

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

Solution Matrix Form
$$\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 18 \\ 2 \end{Bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 18 & 2 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix}} = \frac{(2)(18) - (2)(2)}{(2)(3) - (-1)(2)} \Rightarrow \boxed{x_1 = 4}$$

$$x_2 = \frac{\begin{vmatrix} 3 & 18 \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix}} = \frac{(3)(2) - (-1)(18)}{(2)(3) - (-1)(2)} \Rightarrow \boxed{x_2 = 3}$$

Chapter 9

9.2 Gauss Elimination

For a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Matrix Form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \Rightarrow [A]\{x\} = \{B\}$$

To solve using Gauss Elimination

$$\begin{array}{l} R_1 - \\ R_2 - \\ R_3 - \end{array} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow \text{ Forward Elimination}$$

$$\begin{array}{l} R_1 - \\ R_2' - \\ R_3'' - \end{array} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}' & a_{23}' & b_2' \\ 0 & 0 & a_{33}'' & b_3'' \end{array}\right]$$

Using row operations

$$R_2' = R_2 - \frac{a_{21}}{a_{11}} R_1$$

$$R_3' = R_3 - \frac{a_{31}}{a_{11}} R_1$$

$$R_3'' = R_3' - \frac{a_{32}'}{a_{22}'} R_2'$$

Then, the systems becomes

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2' \\ b_3'' \end{Bmatrix}$$

Backward substitution

$$x_3 = \frac{b_3''}{a_{33}''}$$

$$x_2 = \frac{b_2' - a_{23}\,x_3}{a_{22}'}$$

$$x_3 = \frac{b_1 - a_{12}\,x_2 - a_{13}\,x_3}{a_{11}}$$

Summary

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a_{22}' & a_{23}' & b_2' \\ & & a_{33}'' & b_3'' \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = b_3''/a_{33}''$$
$$x_2 = (b_2' - a_{23}' x_3)/a_{22}'$$
$$x_1 = (b_1 - a_{12} x_2 - a_{13} x_3)/a_{11}$$

Back substitution

Example Solve the following system using Gauss Elimination

$$x_1 + x_2 + x_3 = 6$$
$$2x_1 + 3x_2 + 4x_3 = 20$$
$$3x_1 + 4x_2 + 2x_3 = 17$$

Solution

matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 20 \\ 17 \end{Bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & 20 \\ 3 & 4 & 2 & 17 \end{array}\right] \Rightarrow \text{Forward Elimination}$$

$$\begin{array}{c} R_1 \leftarrow \\ R_2 \leftarrow \\ R_3 \leftarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & 20 \\ 3 & 4 & 2 & 17 \end{array}\right]$$

Row operations (2nd row)

$$R_2' = R_2 - \frac{a_{21}}{a_{11}} R_1 \quad , \; a_{21} = 2, \; a_{11} = 1$$

$$\begin{array}{c} R_1 - \\ R_2' - \\ R_3 - \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 3 & 4 & 2 & 17 \end{array}\right]$$

(2) $\Rightarrow$ $R_2' = 2 - \frac{2}{1} 1 = 0$

(3) $\Rightarrow$ $R_2' = 3 - \frac{2}{1}(1) = 1$

(4) $\Rightarrow$ $R_2' = 4 - \frac{2}{1}(1) = 2$

(20) $\Rightarrow$ $R_2' = 20 - \frac{2}{1}(6) = 8$

3rd row (R_3')

$$R_3' = R_3 - \frac{a_{31}}{a_{11}} R_1 \quad , \; a_{31} = 3, \; a_{11} = 1$$

$$\begin{array}{c} R_1 \\ R_2' \\ R_3' \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

(3) $\Rightarrow$ $R_3' = 3 - \frac{3}{1}(1) = 0$

(4) $\Rightarrow$ $R_3' = 4 - \frac{3}{1}(1) = 1$

(2) $\Rightarrow$ $R_3' = 2 - \frac{3}{1}(1) = -1$

(17) $\Rightarrow$ $R_3' = 17 - \frac{3}{1}(6) = -1$

3rd row (R_3'')

$$R_3'' = R_3' - \frac{a_{32}'}{a_{22}'} R_2' \quad , \; a_{32}' = 1, \; a_{22}' = 1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

(0) $\Rightarrow$ $R_3'' = 0 - \frac{1}{1}(0) = 0$

(1) $\Rightarrow$ $R_3'' = 1 - \frac{1}{1}(1) = 0$

(-1) $\Rightarrow$ $R_3'' = -1 - \frac{1}{1}(2) = -3$

(-1) $\Rightarrow$ $R_3'' = -1 - \frac{1}{1}(8) = -9$

So, the system becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 8 \\ -9 \end{Bmatrix}$$

Backword substitution

$$x_3 = \frac{-9}{-3} \Rightarrow \boxed{x_3 = 3}$$

$$x_2 + 2x_3 = 8 \Rightarrow x_2 = 8 - 2x_3 = 8 - (2)(3) \Rightarrow \boxed{x_2 = 2}$$

$$x_1 + x_2 + x_3 = 6 \Rightarrow x_1 = 6 - x_2 - x_3 = 6 - 2 - 3 \Rightarrow \boxed{x_1 = 1}$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = b''_3/a''_{33}$$

$$x_2 = (b'_2 - a'_{23}x_3)/a'_{22}$$

$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

Back substitution

Pivoting In Gauss Elimination

For the system below,

$$2x_2 + 3x_3 = 6$$
$$4x_1 + 6x_2 + 7x_3 = -3$$
$$2x_1 + x_2 + 6x_3 = 5$$

Solve using Gauss Elimination

$$\begin{bmatrix} 0 & 2 & 3 \\ 4 & 6 & 7 \\ 2 & 1 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -3 \\ 5 \end{Bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 3 & 6 \\ 4 & 6 & 7 & -3 \\ 2 & 1 & 6 & 5 \end{array}\right]$$

For row operations

$$R_2' = R_2 - \frac{a_{21}}{a_{11}} R_1 \Rightarrow 4 \Rightarrow R_2' = 4 - \frac{4}{0}(0)$$

division by Zero

- what to do? move the row with largest value of column to the top row

- Here the row with largest value of column 1 is 2nd row (4, 6, 7, -3)

So,

$$\left[\begin{array}{ccc|c} 4 & 6 & 7 & -3 \\ 0 & 2 & 3 & 6 \\ 2 & 1 & 6 & 5 \end{array}\right]$$

$$4x_1 + 6x_2 + 7x_3 = -3$$
$$2x_2 + 3x_3 = 6$$
$$2x_1 + x_2 + 6x_3 = 5$$

then continue your solution starting from row operations of row (R_3).

practice

Gauss - Jordan Elimination method

$$a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 = b_1$$
$$a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 = b_2$$
$$a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 = b_3$$

Gauss - Jordan

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Elimination $\downarrow$ and $\uparrow$ and normalization

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1^* \\ 0 & 1 & 0 & b_2^* \\ 0 & 0 & 1 & b_3^* \end{array}\right]$$

$$\Rightarrow \left\{\begin{array}{l} x_1 = b_1^* \\ x_2 = b_2^* \\ x_3 = b_3^* \end{array}\right\}$$

Solution

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$