1.4 Modeling and energy methods

For a system with a kinetic energy (T) and potential energy (U), the Lagrangian of this system (L) is defined as $L = T - U$. Therefore, the equation of motion of this system is given by:-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Lagrange Equation

$q, \dot{q} = x, \dot{x}$ spring-mass

$q, \dot{q} = \theta, \dot{\theta}$ Pendulum, Torsional syste.

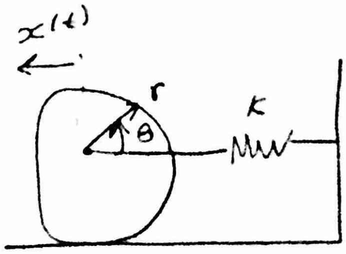
- For this system, the natural frequency can be obtain from

$$T_{max} = U_{max}$$

Example

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{J}{r^2} \dot{x}^2$$

$$U = \frac{1}{2} k x^2$$



- Find nat Frequency of this system

Solution

At Nat. Freq

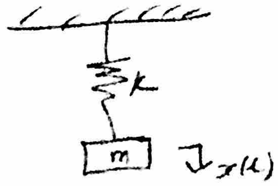
$$T_{max} = U_{max} \Rightarrow \frac{1}{2} m \dot{x}_{max}^2 + \frac{1}{2} \frac{J}{r^2} \dot{x}_{max}^2 = \frac{1}{2} k x_{max}^2$$

$$x(t) = X \sin(\omega_n t + \phi) \rightarrow x(t)_{max} = X$$

$$\dot{x}(t) = \omega_n X \cos(\omega_n t + \phi) \Rightarrow \dot{x}(t) = V = \omega_n X$$

$$\frac{1}{2} m (\omega_n X)^2 + \frac{1}{2} \frac{J}{r^2} (\omega_n X)^2 = \frac{1}{2} k (X^2)$$

$$\Rightarrow \omega_n^2 = \frac{\frac{1}{2} k}{\frac{1}{2} m + \frac{1}{2} \frac{J}{r^2}} \Rightarrow \omega_n = \sqrt{\frac{k}{m + J/r^2}}$$



Example: Spring-mass system

$$T = \frac{1}{2} m \dot{x}^2$$

$$U = \frac{1}{2} k x^2$$

Find Eq. of motion

Solution

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad , \quad q = x, \quad \dot{q} = \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Rightarrow \frac{d}{dt} = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\Rightarrow \quad m\ddot{x} - (-kx) = 0 \quad \Rightarrow \quad m\ddot{x} + kx = 0$$

Nat. Freq

$$x_{max} = X$$

$$\dot{x}_{max} = \omega_n X$$

$$T_{max} = U_{max} \quad \Rightarrow \quad \cancel{\frac{1}{2}} m \dot{x}_{max}^2 = \cancel{\frac{1}{2}} k x_{max}^2$$

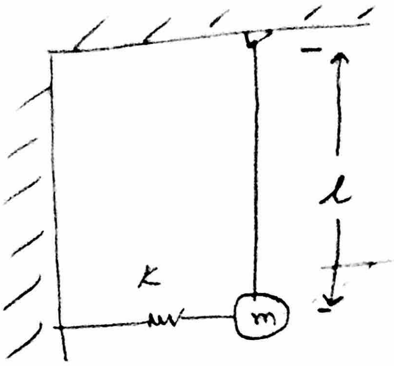
$$m \omega_n^2 \cancel{X^2} = k \cancel{X^2} \quad \Rightarrow \quad \omega_n^2 = \frac{k}{m} \quad \Rightarrow \quad \omega_n = \sqrt{\frac{k}{m}}$$

Example

$$T = \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = \frac{1}{2} k x^2 + mgh$$

Find Equation of motion

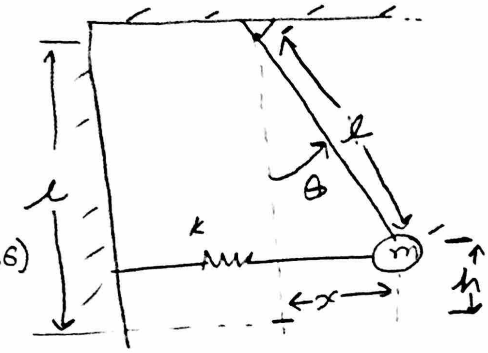


Solution

$$h = l - l\cos\theta$$
$$h = l(1 - \cos\theta)$$

$$x = l\sin\theta$$

$$\Rightarrow U = \frac{1}{2} k l^2 \sin^2\theta + mgl(1-\cos\theta)$$



$$L = T - U$$
$$= \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} k l^2 \sin^2\theta - mgl(1-\cos\theta)$$

$= mgl + mgl\cos\theta$

Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \Rightarrow \frac{d}{dt} = m l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -k l^2 \sin\theta\cos\theta - mgl\sin\theta$$

$$m \cancel{l^2} \ddot{\theta} - \left(-k \cancel{l^2} \sin\theta\cos\theta - mg\cancel{l}\sin\theta\right) = 0$$

$$m l \ddot{\theta} + k l \sin\theta\cos\theta + mg\sin\theta = 0$$

For small $\theta \Rightarrow \sin\theta = \theta$, $\cos\theta = 1$

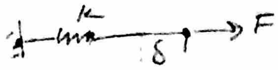
$$m l \ddot{\theta} + k l \theta + mg\theta = 0$$

$$\Rightarrow \ddot{\theta} + \theta\frac{(kl + mg)}{ml} = 0$$

$$\omega_n^2 = \frac{kl + mg}{ml} \Rightarrow \omega_n = \sqrt{\frac{kl+mg}{ml}}$$

* Stiffness

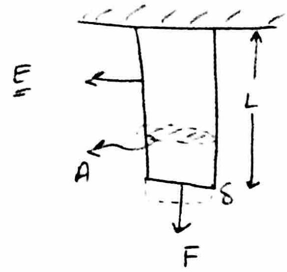
For a spring $\Rightarrow$ Stiffness (k) (N/m)



$\rightarrow$ $\text{stiffness} = \dfrac{\text{Load}}{\text{Deflection}}$, $k = \dfrac{F}{\delta}$

- Bar - Axial load

From Mechanics of materials

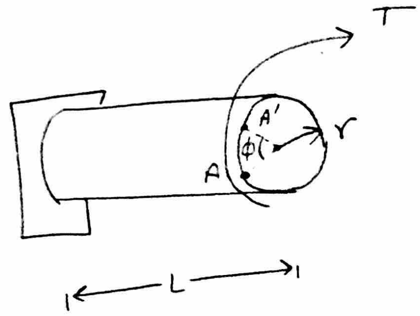


$$\delta = \frac{FL}{AE}$$

Load $\rightarrow F$
Deflection $\rightarrow \delta$

$$K = \frac{\text{Load}}{\text{Deflec}} = \frac{F}{\delta} = \frac{\cancel{F}}{\cancel{F}L/AE} \Rightarrow K = \frac{AE}{L}$$

- Bar with torsion



ϕ = Angle of twist

$$\phi = \frac{TL}{JG}$$

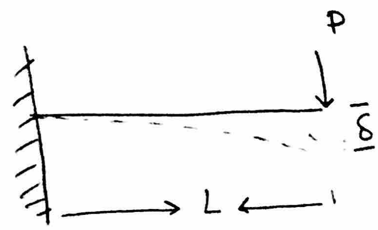
J : Polar moment of Inertia
G : Shear modulus

Load $\rightarrow T$
Deflection $\rightarrow \phi$

$$\frac{\text{Load}}{\text{Deflection}} = \frac{T}{\phi} = \frac{\cancel{T}}{\cancel{T}L/JG} \Rightarrow K_t = \frac{JG}{L}$$

torsonal

- Cantilever Beam

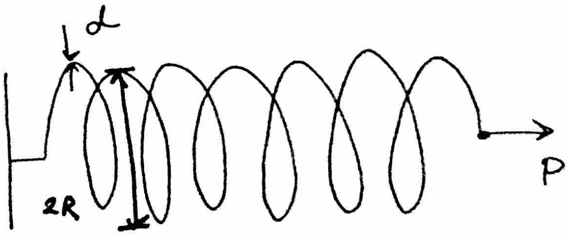


$$\delta = \frac{PL^3}{3EI}$$

$$K = \frac{\text{Load}}{\text{Def}} = \frac{P}{PL^3/3EI} \Rightarrow k = \frac{3EI}{L^3}$$

- Helical spring

$$K = \frac{Gd^4}{64nR^3}$$



n: number of turns
R: turn radius
G: Shear Modulus

Springs in Series

→ Different deflections

Equivalent Stiffness

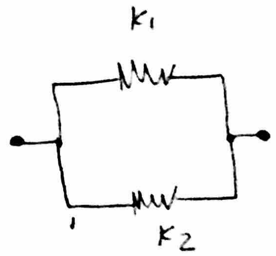
k_{eq} ⇒

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

For n springs

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_n} \Rightarrow k_{eq} = \frac{(k_1)(k_2)\dots(k_n)}{k_1 + k_2 + \dots + k_n}$$

Springs in parallel

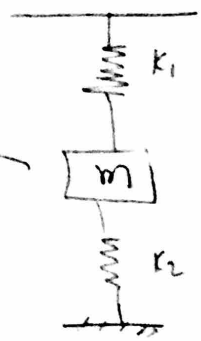


$$k_{eq} = k_1 + k_2$$

For n springs

$$k_{eq} = k_1 + k_2 + \dots + k_n$$

Same deflection

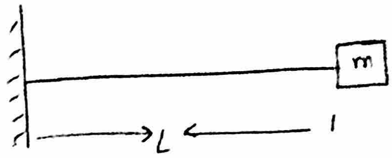


Parallel (same deflection)

$$k_{eq} = k_1 + k_2$$

Example

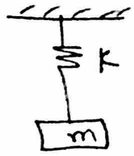
$m = 1000\,kg$, $L = 2\,m$
$I = 5.2 \times 10^{-4}\,m^4$, $E = 69\,GPa$



Find Natural Freq

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{mL^3}} = \sqrt{\frac{(3)(69\times10^9)(5.2\times10^{-5})}{(1000)(2)^3}} \Rightarrow \omega_n = 116\ rad/sec$$

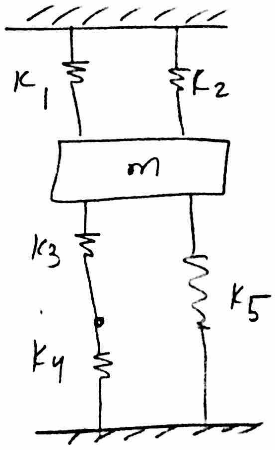
$$K = \frac{3EI}{L^3}$$



Example

(1) Find K_{eq}

(2) ω_n



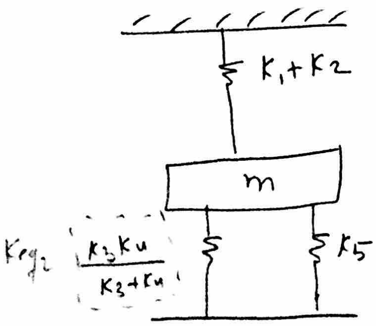
Solution

K_1 and K_2 = parallel

$K_{eq1} = K_1 + K_2$

K_3 and K_4 = Series

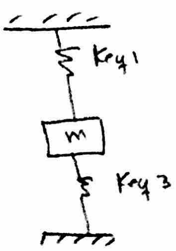
$$K_{eq2} = \frac{K_3 K_4}{K_3 + K_4}$$



$\Rightarrow$ K_{eq2} and K_5 = Parallel

$$K_{eq3} = K_{eq2} + K_5 = \frac{K_3 K_4}{K_3 + K_4} + K_5$$

$\Rightarrow$

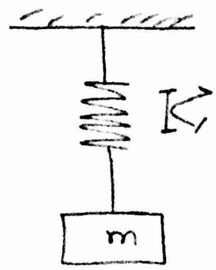


$\Rightarrow$ K_{eq1} and K_{eq3} are in parallel (both have same deflection)

$$K_{eq4} = K_{eq1} + K_{eq3}$$

$$K = K_1 + K_2 + \left(\frac{K_3 \cdot K_4}{K_3 + K_4}\right) + K_5$$

$$K_{eq} = K_1 + K_2 + \left(\frac{K_3 \cdot K_4}{K_3 + K_4}\right) + K_5$$



$$\Rightarrow \quad \omega_n = \sqrt{\frac{K_{eq}}{m}}$$

If $m = 10$ kg, Find ω_n for

(a) $K_1 = 1000$ N/m , $K_2 = 3000$ N/m

$K_3 = K_4 = K_5 = 0$

(b) $K_3 = 1000$ N/m , $K_4 = 3000$ N/m

$K_1 = K_2 = K_5 = 0$

(a) $K_{eq} = 1000 + 3000 + 0 \Rightarrow K_{eq} = 4000$ N/m

$$\omega_n = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

(b) $K_{eq} = 0 + 0 + \frac{(1000)(3000)}{1000 + 3000} + 0$

$K_{eq} = 750$ N/m

$$\omega_n = \sqrt{\frac{750}{10}} = 8.66 \text{ rad/s}$$

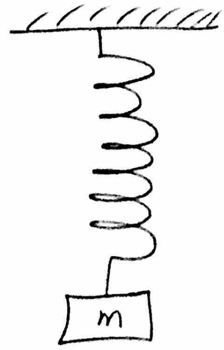
Example

$m = 1$ kg $\quad G = 10$ GPa

$R = 5$ mm , $n = 1000$ turn

$\omega_n = 25$ rad/s

Find d



$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = \omega_n^2 m$$

$$k = \frac{Gd^4}{64 n R^3} \Rightarrow d = \sqrt[4]{\frac{64(n)(k)R^3}{G}} = \sqrt[4]{\frac{64(n)(\omega_n)^2(m)(R^3)}{G}}$$

$$d = \sqrt[4]{\frac{(64)(1000)(25)^2(1000)(0.005)^3}{10 \times 10^9}} \Rightarrow d = 8.8 \text{ cm}$$

Example 1

, undamped system

$X = 1$ mm , $\Phi = 2$ rad , $w_n = 5$ rad/sec

find Initial conditions $x(0)$ and $\dot{x}(0)$ | $x(0) = x_0$, $\dot{x}(0) = v_0$

Solution

$$x(t) = X \sin(w_n t + \phi)$$

$$X = \sqrt{(x_0)^2 + \left(\frac{v_0}{w_n}\right)^2} \Rightarrow 1 = \sqrt{x_0^2 + \frac{v_0^2}{25}}$$

$$\Rightarrow \quad 1 = x_0^2 + \frac{v_0^2}{25} \quad — (1)$$

$$\phi = \tan^{-1}\left(\frac{x_0}{v_0} w_n\right) \Rightarrow \tan\phi = \frac{x_0}{v_0} w_n$$

$$\tan 2\text{rad} = \frac{x_0}{v_0}(5) = -2.185$$

$$\Rightarrow x_0 = -0.437 v_0$$ – substitute in (1)

$$1 = (-0.437)^2 v_0^2 + v_0^2/25$$

$$1 = 0.19 v_0^2 + 0.04 v_0^2$$

$$v_0^2 = \frac{1}{0.23} \Rightarrow$$

$$v_0 = \pm 2.08 \text{ mm/s}$$

$$x_0 = \mp 0.909 \text{ mm}$$

OR

$$x(t) = X \sin(w_n t + \phi)$$

$t = 0$

$$x(0) = x_0 = X \sin\phi \Rightarrow x_0 = (1)\sin(2)$$

$$x_0 = 0.909 \text{ mm}$$

$$\dot{x}(t) = w_n X \cos(w_n t + \phi)$$

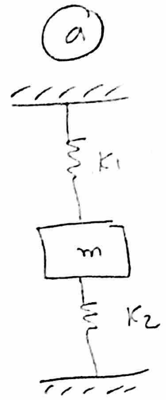
$t = 0$

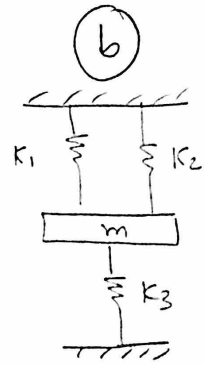
$$\dot{x}(0) = v_0 = w_n X \cos\phi = (5)(1)\cos(2)$$

$$v_0 = -2.08 \text{ mm/s}$$

Example: 2

Find ω_n for System (a) and (b)

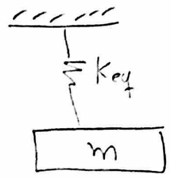




Solution

(a) Springs in parallel

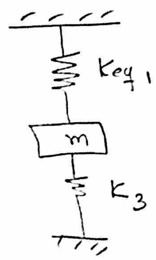
$$K_{eq} = K_1 + K_2$$



$$\omega_n = \sqrt{\frac{K_1 + K_2}{m}}$$

(b) Spring $K_1 + K_2$ Parallel

$$K_{eq1} = K_1 + K_2$$



$\Rightarrow$ K_{eq1} and K_3

parallel

$$K = K_1 + K_2 + K_3$$

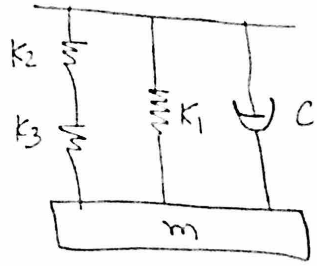
$$\omega_n = \sqrt{\frac{K_1 + K_2 + K_3}{m}}$$

Example 5

$m = 10\ kg$, $k_1 = 4000\ N/m$

$k_2 = 200\ N/m$, $k_3 = 1000\ N/m$

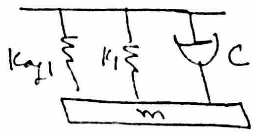
$C = 100\ kg/s$



Find ① ω_n
② ζ

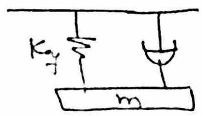
Solution

(a) k_2 and $k_3 \Rightarrow$ Series $\Rightarrow k_{eq1} = \dfrac{k_2 \cdot k_3}{k_2 + k_3}$



k_{eq1} and $k_1 \Rightarrow$ Paralle $\Rightarrow k_{eq} = k_1 + \dfrac{k_2 \cdot k_3}{k_2 + k_3}$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 + \frac{k_3 \cdot k_2}{k_2 + k_u}}{m}}$$



$$= \sqrt{\frac{4000 + \left(\frac{(200)(1000)}{200 + 1000}\right)}{10}} \Rightarrow \omega_n = 20.41\ rad/s$$

(b) $m\ddot{x} + c\dot{x} + kx = 0$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

$$\frac{c}{m} = 2\zeta\omega_n \Rightarrow \zeta = \frac{c}{2m\omega_n} = \frac{100}{(2)(10)(20.41)}$$

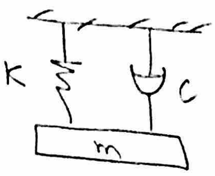
$\boxed{\zeta = 0.\cancel{0}245}$ underdamped

0.245

Example 4

$m = 750\ kg,\ \zeta = 0.25,\ f_n = 15\ Hz$

Find C and K



Solution

(1) $m\ddot{x} + C\dot{x} + Kx = 0$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

$2\zeta\omega_n = \frac{C}{m} \Rightarrow C = 2\zeta\omega_n m$, $\omega_n = 2\pi f_n$

$C = 2(0.25)(2\pi)(15)(750)$

$C = 35\,343\ kg/s$

$K \Rightarrow \omega_n = \sqrt{\frac{K}{m}} \Rightarrow \omega_n^2 = \frac{K}{m}$

$K = \omega_n^2 m = (2\pi \times 15)^2\ 750$

$K = 7.281\ kN/m$