Chapter 6 Roots of Equations: Open Methods

⊛ Simple-fixed point Iterations method

For a function $f(x)$, Simple fixed point method can be used to obtain an approximate root following the procedure:

① Convert $f(x)$ to the form of $x = g(x)$, where $|g'(x)| < 1$.

* For example $f(x) = x^2 - 2x + 3$, write $f(x)$ In form of $x = g(x)$. If $x > 2$

- $f(x) = 0 \Rightarrow x^2 - 2x + 3 = 0$

① $x = \boxed{\frac{x^2+3}{2}}$ $g(x)$

$g(x) = \frac{x^2+3}{2}$

-Check $|g'(x)| < 1$

$g'(x) = x$, No

Condition Not satisfied

② $x = \boxed{\sqrt{2x-3}}$ $g(x)$

$g(x) = \sqrt{2x-3}$

-check $|g'(x)| < 1$

$g'(x) = \frac{1}{\sqrt{2x-3}}$

$|g'(x)| < 1$ for all x

Condition satisfied

Then $g(x) = \sqrt{2x-3}$ $x = g(x)$

② Do iterations $x_{i+1} = g(x_i)$ $i = 0, 1, 2, \ldots$

Example $f(x) = e^{-x} - x$ use simple fixed-point iteration method to find the root of $f(x)$ $x > 0$

Use $x_0 = 1$ and $\varepsilon_a < 15\%$.

Solution

① convert to $x = g(x)$

$\Rightarrow f(x) = 0 \Rightarrow e^{-x} - x = 0 \Rightarrow x = e^{-x}$

$x = g(x)$, $g(x) = e^{-x}$

- check $|g'(x)| < 1$

$g'(x) = -e^{-x} = \frac{-1}{e^x}$

$\left|\frac{-1}{e^x}\right| < 1$ always for $x > 1$

② Iterations $x_{i+1} = g(x_i)$, $i = 0, 1, 2, \ldots$

$i = 0$ $x_1 = g(x_0) \Rightarrow x_1 = g(1) = e^{-1} \Rightarrow x_1 = 0.3679$

$i = 1$ $x_2 = g(x_1) = g(0.3679) = e^{-0.3679} \Rightarrow x_2 = 0.6922$

$i = 2$ $x_3 = g(x_2) = g(0.6922) = e^{-0.6922} \Rightarrow x_3 = 0.5004$

i	x_{i+1}	ε_a
0	$x_1 = 0.3679$	N/A
1	$x_2 = 0.6922$	$\varepsilon_a = \left\lvert\frac{0.6922 - 0.3679}{0.6922}\right\rvert 100\% = 46.9\%$
2	$x_3 = 0.5004$	$\varepsilon_a = \left\lvert\frac{0.5004 - 0.6922}{0.5004}\right\rvert 100\% = 38.3\%$
3	$x_4 = 0.6062$	$\varepsilon_a = \left\lvert\frac{0.6062 - 0.5004}{0.6062}\right\rvert 100\% = 17.4\%$
4	$x_5 = 0.5454$	$\varepsilon_a = 11.2\%$

$$\varepsilon_a = \left|\frac{\text{Present } x_{i+1} - \text{Previous } x_{i+1}}{\text{Present } x_{i+1}}\right| 100\%$$

$$\boxed{x_{i+1} = x_5 = 0.5454}$$

$i=3$ $\quad x_4 = g(x_3) = g(0.5004) = e^{-0.5004} \Rightarrow x_4 = 0.6062$

$i=4$ $\quad x_5 = g(x_4) = g(0.6062) = e^{-0.6062} \Rightarrow x_5 = 0.5454$

$$\epsilon_a = \left| \frac{0.5454 - 0.6062}{0.5454} \right| 100\% = 11.2\%$$

- Practice : Continue your iterations to reach

$\epsilon_a < 2\%$

Chapter 6

⊛ Newton-Raphson (NR) method

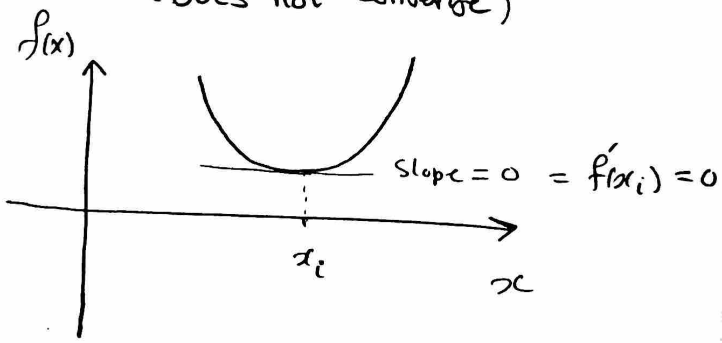
we can find approximate root of function $f(x)$ with an initial guess x_i, as:

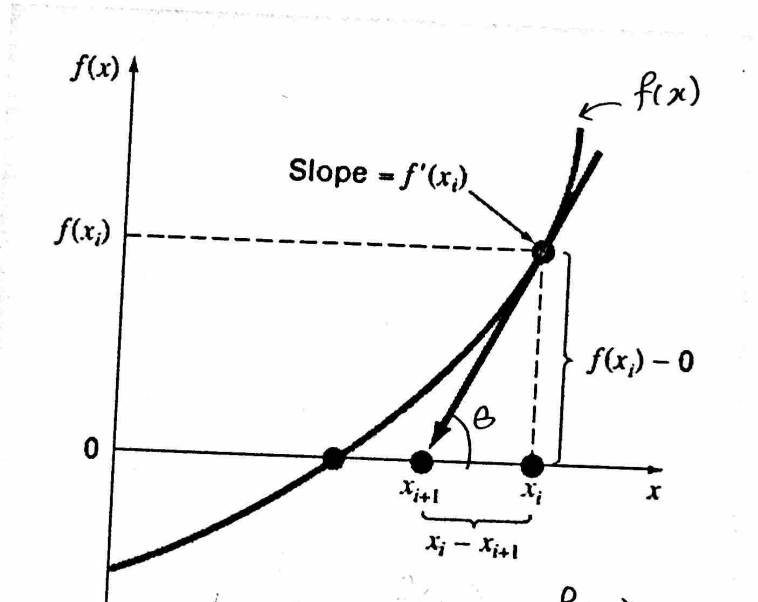
$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}} \qquad i = 0, 1, 2, \ldots$$

If $f'(x_i) = 0$ (zero slope) $\Rightarrow x_{i+1} = \infty$

we can't use NR

(Does not converge)





$$\text{Slope} = \tan\theta = f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

__Example:__ use NR to find root of $f(x) = e^{-x} - x$

$x_0 = 0$, $\varepsilon_a < 1\%$

__Solution__

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad f'(x) = -e^{-x} - 1$$

__$i = 0$__

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{1}{-2} \Rightarrow \boxed{x_1 = 0.5}$$

$$f(0) = e^{-0} - 0 = 1$$

$$f'(0) = -e^{-0} - 1 = -2$$

$\underline{i=1}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5 - \frac{f(0.5)}{f'(0.5)} \Rightarrow x_2 = 0.5663$$

$$f(0.5) = e^{-0.5} - 0.5 = 0.1065$$

$$f'(0.5) = -e^{-0.5} - 1 = -1.6065$$

$\underline{i=2}$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5663 - \frac{f(0.5663)}{f'(0.5663)} \Rightarrow x_3 = 0.5671$$

i	x_{i+1}	ϵ_a
0	$x_1 = 0.5$	N/A
1	$x_2 = 0.5663$	$\epsilon_a = \left\lvert \frac{0.5663 - 0.5}{0.5663} \right\rvert 100\% = 11.7\%$
2	$x_3 = 0.5671$	$\epsilon_a = \left\lvert \frac{0.5671 - 0.5663}{0.5671} \right\rvert 100\% = 0.15\%$

$$x_{i+1} = x_3 = 0.5671$$

$$\epsilon_a = \left\lvert \frac{\text{Present } x_{i+1} - \text{Previous } x_{i+1}}{\text{Present } x_{i+1}} \right\rvert 100\%$$

NR Much faster than Simple fixed point method

- Secant method

To find a root of function $f(x)$ using secant method:

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}, \quad i = 0, 1, 2, \ldots$$

x_{i-1} and x_i are initial guesses.

Remember False Pos. meth.

$$x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}$$

Example: $f(x) = e^{-x} - x$ use secant method to estimate root. Given $x_{-1} = 0$, $x_0 = 1$ for $\epsilon_a < 5\%$

Solution

$i = 0$

$i = 0$
$x_i = x_0$
$x_{i-1} = x_{-1}$
$x_{i+1} = x_1$

$$x_1 = x_0 - \frac{f(x_0)(x_{-1} - x_0)}{f(x_{-1}) - f(x_0)}$$

$$= 1 - \frac{f(1)(0-1)}{f(0) - f(1)}$$

$$x_1 = 0.6127$$

$\epsilon_a = N/A$

$i = 1$

$$x_2 = x_1 - \frac{f(x_1)(x_0 - x_1)}{f(x_0) - f(x_1)}$$

$$= 0.6127 - \frac{f(0.6127)(1 - 0.6127)}{f(1) - f(0.6127)}$$

$$\Rightarrow x_2 = 0.5638$$

$$\epsilon_a = \left| \frac{\text{Present } x_{i+1} - \text{Previous } x_{i+1}}{\text{present } x_{i+1}} \right| 100\% = \left| \frac{0.5638 - 0.6127}{0.5638} \right| 100\% = 8.6\%$$

$\underline{i = 2}$

$$x_3 = x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)}$$

$$= 0.5638 - \frac{f(0.5638)(0.6127 - 0.5638)}{f(0.6127) - f(0.5638)}$$

$$x_3 = 0.5672$$

$$\epsilon_a = \left| \frac{0.5672 - 0.5638}{0.5672} \right| 100\% = 0.58\%$$

Root $\boxed{x_3 = 0.5672}$

6.3.3 Modified Secant method

Root can be estimated from

$$x_{i+1} = x_i - \frac{\delta x_i \, f(x_i)}{f(x_i + \delta x_i) - f(x_i)} \qquad i = 0, 1, 2, \ldots$$

$$\delta x_i = x_i - x_{i-1}$$

- See example 6.8 page 161 From text book

6.5 Multiple Roots (Repeated Roots)

For a function

$$f(x) = x^3 - 5x^2 + 7x - 3$$

$$f(x) = (x-3)(x-1)(x-1) \quad , \text{ Roots } \quad x_1 = 3, \quad x_{2,3} = 1$$

$x_{2,3} = 1$ → Double or multiple root (repeated root)

In case of multiple roots, we use Modified Newton–Raphson method, as:

$$x_{i+1} = x_i - \frac{f(x_i)\, f''(x_i)}{[f'(x_i)]^2 - f(x_i)\, f''(x_i)} \qquad i = 0, 1, 2, \dots$$

Example :- $f(x) = x^3 - 5x^2 + 7x - 3$

Use modified NR to find root $x_0 = 0$, $\varepsilon_a < 5\%$

Solution

$i = 0$

$$x_1 = x_0 - \frac{f(x_0)\, f''(x_0)}{[f'(x_0)]^2 - f(x_0)\, f''(x_0)}$$

$$= 0 - \frac{f(0)\, f''(0)}{[f'(0)]^2 - f(0) f''(0)} = 1.1053$$

$f'(x) = 3x^2 - 10x + 7$, $f'(0) = 7$

$f''(x) = 6x - 10$, $f''(0) = -10$

$f(0) = -3$

$\varepsilon_a = N/A$

$\underline{i = 1}$

$$x_2 = x_1 - \frac{f(x_1)\, f''(x_1)}{[f'(x_1)]^2 - f(x_1) f''(x_1)}$$

$$= 1.1053 - \frac{f(1.1053)\, f''(1.1053)}{[f'(1.1053)]^2 - f(1.1053) f''(1.1053)}$$

$$x_2 = 1.0031$$

$$\epsilon_a = \left| \frac{\text{Present } x_{i+1} - \text{Previous } x_{i+1}}{\text{Present } x_{i+1}} \right| 100\%$$

$$= \left| \frac{1.0031 - 1.1053}{1.0031} \right| \times 100\% = 10.2\%$$

$\underline{i = 2}$

$$x_3 = x_2 - \frac{f(x_2)\, f''(x_2)}{[f'(x_2)]^2 - f(x_2) f''(x_2)} \qquad x_2 = 1.0031$$

$$x_3 = 1.0000$$

$$\epsilon_a = \left| \frac{1 - 1.0031}{1} \right| 100\% = 0.31\%$$

Root $\boxed{x_3 = 1.000}$