Chapter One: Introduction to Vibration and Free Response

- What is vibration?

  It is the study of repetitive motion caused by the inertia (mass) and the flexibility (stiffness) of the system

  Mass → Kinetic energy

  Stiffness → Potential Energy

- Examples of vibration
  1. Pendulum
  2. Car suspension
  3. Strings in musical devices

- Basic elements of a vibrating system
  1. Mass (m) → Kinetic energy storage
  2. Stiffness (k) → Potential Energy storage
  3. Damping (c) → Energy loss
  4. Force (F) → Applied external forces

* Types of vibration

  → Single Degree-of-Freedom (SDOF)

  → Multi-degrees-of-freedom (MDOF)

  → Free Vibration (No Forces)
    - → Undamped (No damping)
    - → Damped

  → SDOF

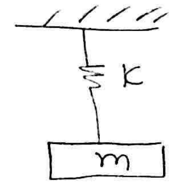
  → MDOF

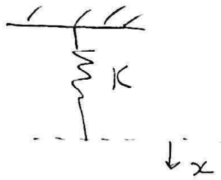
  → Forced Vibration (with external applied forces)
    - → undamped (No damping)
    - → Damped (with damping)

- Spring-mass model

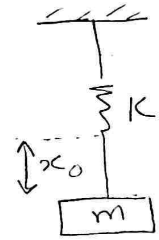
Spring-mass model is the most common model used in the study of vibration - SDOF

Spring constant (K [N/m])
mass (m kg)





$x=0$ ⇒ Add mass(m)

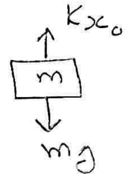


one degree of freedom (x)

Free-body diagram

Static

$\uparrow \sum F = 0$



$\Rightarrow x_0 =$

$Kx_0 - mg = 0$

$\Rightarrow x_0 = \frac{mg}{K}$ [m]

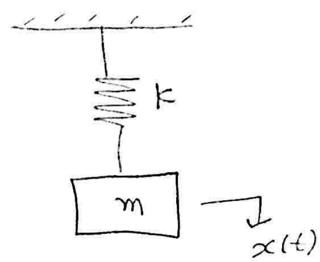
Static Deflection

Free
* Vibration of Spring-mass system

Dymanic

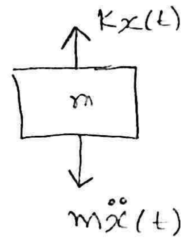
$\uparrow \sum F = ma = m\ddot{x}$ (accelaration)

$m\ddot{x} + Kx = 0$ Equation of motion



Divide by m

$$\ddot{x} + \frac{K}{m}x = 0 \Rightarrow \ddot{x} + \omega_n^2 x = 0$$



$$\omega_n^2 = \frac{K}{m} \Rightarrow \omega_n = \sqrt{\frac{K}{m}}$$

Natural Frequency of the system (rad/sec)

- Another models for SDOF Vibration systems

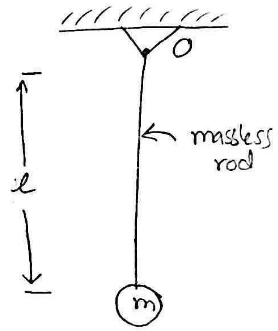
1) Pendulum

$$\sum M_O = J\ddot{\theta} \rightarrow \text{Angular acceleration}$$

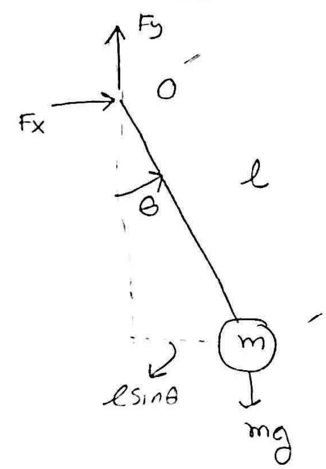
Moment about O

mass moment of Inertia

$J = ml^2$



F.B.D



$\xrightarrow{}$ clockwise +

$$-mgl\sin\theta = ml^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\theta(t) = 0$$

$$\omega_n = \sqrt{\frac{g}{l}}$$

$$= \sqrt{\frac{m/s^2}{m}} = \sqrt{\frac{1}{s^2}}$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{N/m}{kg}} \quad N = kg \cdot m/s^2$$

$$= \sqrt{\frac{kg\,\frac{m}{s^2}/m}{kg}} = \sqrt{\frac{1}{s^2}} =$$

2) Torsional system

$$\sum T_O = J\ddot{\theta}$$

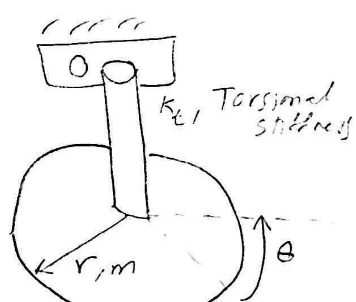
$$-K_t\theta = J\ddot{\theta}$$

$$\Rightarrow J\ddot{\theta} + K_t\theta = 0$$

$$\ddot{\theta} + \frac{K_t}{J}\theta = 0$$

$$\omega_n = \sqrt{\frac{K_t}{J}} = \sqrt{\frac{N.m}{kg\,m^2}} = \sqrt{\frac{kg.\frac{m}{s^2}.m}{kg.m^2}}$$

$$= \sqrt{1/s^2}$$



$J = mr^2$

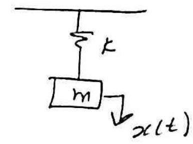
mass moment of Inertia ($kg.m^2$)

K_t ($\frac{N.m}{rad}$)

Free-undamped vibrati

$\Rightarrow \quad \ddot{x} + \omega_n^2 x = 0$, 2nd order Differential Equati (ODE)

Soluti , we need to find $x(t)$



$x(t) = A e^{\lambda t}$

Velocity ← $\dot{x}(t) = \lambda A e^{\lambda t}$

Accel ← $\ddot{x}(t) = \lambda^2 A e^{\lambda t}$

$\lambda^2 A e^{\lambda t} + \omega_n^2 A e^{\lambda t} = 0 \Rightarrow \lambda^2 + \omega_n^2 = 0$

$\lambda^2 = -\omega_n^2 \Rightarrow \lambda_{1,2} = \pm \hat{i}\omega_n$, $\hat{i} = \sqrt{-1}$ complex number

⇒ From ODE

$x(t) = A_1 e^{\hat{i}\omega_n t} + A_2 e^{-\hat{i}\omega_n t}$

or

$x(t) = A \cos\omega_n t + B \sin\omega_n t$ ⇒ A and B constants ⇒ From initial conditions (IC's)

or

$x(t) = X \sin(\omega_n t + \phi)$

X: Vibration Amplitude
ϕ: phase shift (angle)
} From IC's

How?

ID $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\Rightarrow \sin(\omega_n t + \phi) = \sin\omega_n t \cos\phi + \cos\omega_n t \sin\phi$

$X \sin(\omega_n t + \phi) = X \sin\omega_n t \cos\phi + X \cos\omega_n t \sin\phi$

compare to $x(t) = A\cos\omega_n t + B \sin\omega_n t$

$\frac{A}{B} \Leftarrow \quad A = X \sin\phi, \quad B = X \cos\phi \quad \Rightarrow \quad A^2 = X^2 \sin^2\phi \; + \; B^2 = X^2 \cos^2\phi$

$A^2 + B^2 = X^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{=1}$

$\frac{\sin\phi}{\cos\phi} = \tan\phi = \frac{A}{B}$

$X = \sqrt{A^2 + B^2}$

$\phi = \tan^{-1}\left(\frac{A}{B}\right)$

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If the Initial conditions are

$t=0$ $x(0) = x_0$, what are A, B, X and ϕ?
Initial State $\dot{x}(0) = v_0$

Consider $x(t) = A\cos w_n t + B\sin w_n t$

Substitute IC's in $x(t)$

$x(0) = x_0 = A + 0 \Rightarrow \boxed{A = x_0}$

$\dot{x}(t) = -w_n A \sin w t + w_n B \cos w_n t$

$\dot{x}(0) = v_0 = w_n B \Rightarrow \boxed{B = \frac{v_0}{w_n}}$

$$\Rightarrow \quad X = \sqrt{A^2 + B^2} \Rightarrow \boxed{X = \sqrt{(x_0)^2 + \left(\frac{v_0}{w_n}\right)^2}}$$

$$\phi = \tan^{-1}\left(\frac{A}{B}\right) \Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{x_0}{B_0} w_n\right)}$$

plot $x(t) = X\sin(w_n t + \phi)$

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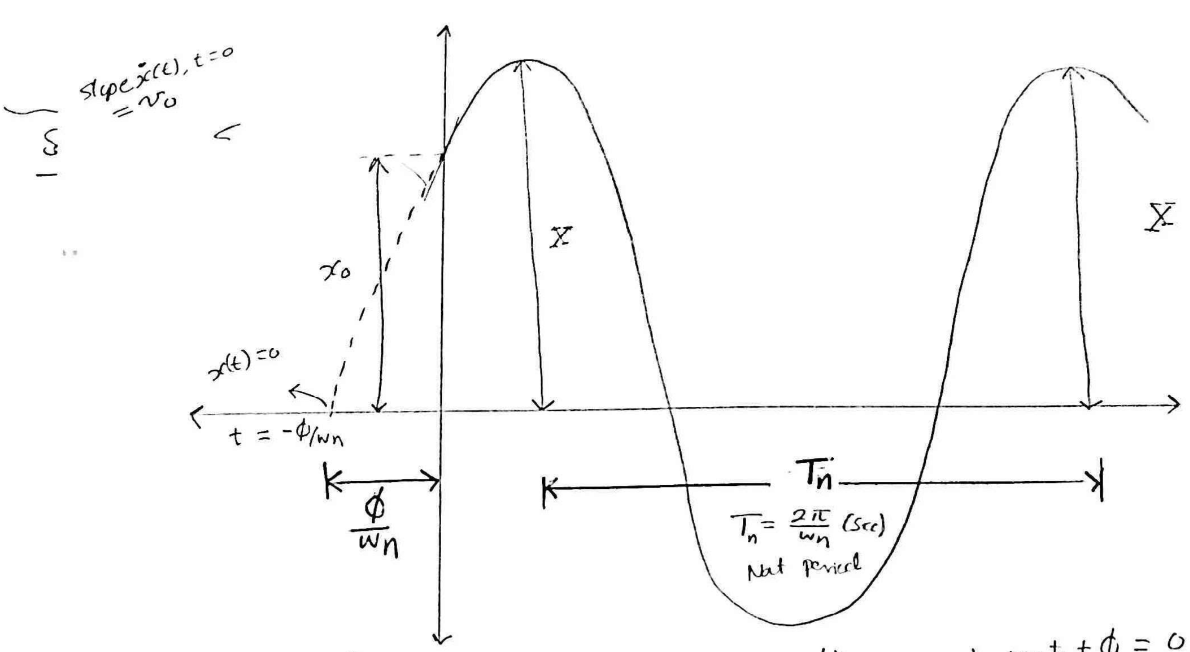
$x(t)$



X = Amplitude

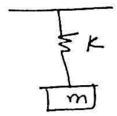
$x(t) = 0 = \overset{\neq 0}{X} \sin(w_n t + \phi) \Rightarrow \sin(w_n t + \phi) = 0 \Rightarrow w_n t + \phi = 0$
$\Rightarrow t = \frac{-\phi}{w_n}$

Nat frequency $(f_n) = 2\pi w_n = \frac{1}{T_n}$ (Hz)

Example 1

$m = 30\ kg$, $f_n = 10\ Hz$

$K = ?$

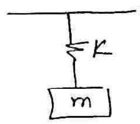


Solution

$$\omega_n = 2\pi f_n = \sqrt{\frac{K}{m}} \Rightarrow (2\pi f_n)^2 = \frac{K}{m} \Rightarrow K = m(2\pi f_n)^2$$

$$K = (30)(2\pi(10))^2 \Rightarrow K = 1.1184 \times 10^5\ N/m$$

Example 2



$m = 2\ kg$, $K = 200\ N/m$

Find X and ϕ

(a) $x_0 = 2\ mm$, $v_0 = 1\ mm/s$

(b) $x_0 = -2\ mm$, $v_0 = 1\ mm/s$

Solution

$$X = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}, \quad \omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{200}{2}} \Rightarrow \omega_n = 10\ rad/s.$$

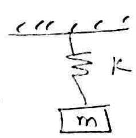
$$\phi = \tan^{-1}\left(\frac{x_0}{v_0} \cdot \omega_n\right)$$

(a), (b) $\Rightarrow X = \sqrt{(2)^2 + \left(\frac{1}{10}\right)^2} \Rightarrow X = 2.0025\ mm$

(a) $\phi = \tan^{-1}\left(\frac{2}{1} \cdot 10\right) \Rightarrow \phi = 87.15\ deg$

(b) $\phi = \tan^{-1}\left(\frac{-2}{1} \cdot 10\right) \Rightarrow \phi = -87.15\ deg.$

Example 3



$m = 49.2\ g$, $K = 857.8\ N/m$

$x_0 = 10\ mm$, $v_0 = 0\ mm/s$

Find

1. Natural Frq (ω_n)
2. Natural period (T_n)
3. Maximum → Displacement amplitude, → Velocity ", → Acceleration "
4. ϕ

Solution

① $\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} \Rightarrow \omega_n = 132\ rad/s$

② $T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = \frac{2\pi}{132} \Rightarrow T_n = 0.0476\ Sec$

③ Disp $x(t) = X \sin(\omega_n t + \phi)$

$X = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} = x_0 \Rightarrow X = 10\ mm$

velocity $\Rightarrow \dot{x}(t) = \underbrace{\omega_n X}_{V\ \text{velocity amp}} \cos(\omega_n t + \phi)$

$V = \omega_n X = (132)(10)$
$= 1320\ mm/s$

Acceleration $\ddot{x}(t) = -\underbrace{\omega_n^2 X}_{A\ \text{accel amplitude}} \sin(\omega_n t + \phi)$

$A = \omega_n^2 X$
$= (132)^2 (10) = 174.24\ m/s^2$

④ $\phi = \tan^{-1}\left(\frac{x_0}{v_0}\omega_n\right) \Rightarrow \phi = \tan^{-1}(\infty)$
$\phi = 90^\circ$

Example

For pendulum system

① $T_n = 3$ seconds , find length (l) , $g = 9.81\ m/s^2$

$$T_n = \frac{2\pi}{\omega_n} \quad , \quad \omega_n = \sqrt{\frac{g}{l}}$$

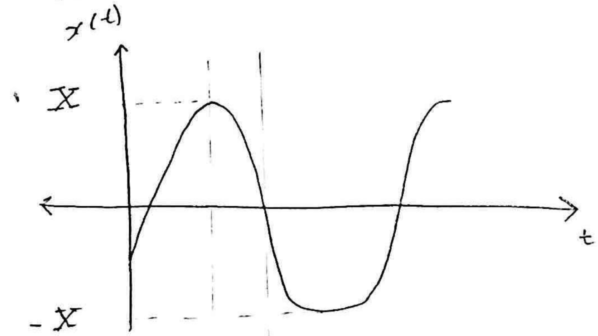
$$\Rightarrow T_n = \frac{2\pi\sqrt{l}}{\sqrt{g}} \Rightarrow l = \frac{g T_n^2}{(2\pi)^2} = \frac{(9.81)(3)}{(2\pi)^2} \Rightarrow l = 2.237\ m$$

- Harmonic Motion

Displacement $\quad x(t) = X \sin(\omega_n t + \phi) = V \cos(\omega_n t + \phi)$

Velocity $\quad \dot{x}(t) = \omega_n X \cos(\omega_n t + \phi) = A \sin(\omega_n t + \phi)$

Acceleration $\quad \ddot{x}(t) = -\omega_n^2 X \sin(\omega_n t + \phi)$



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$x(t) = 0 \rightarrow$ max $\dot{x}(t)$
$\quad \rightarrow \ddot{x}(t) = 0$

$x(t)$ max/min $\rightarrow \dot{x}(t) = 0$
$\quad \rightarrow$ min/max $\ddot{x}(t)$

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