* Roots of Equations

$$f(x) = ax^2 + bx + C$$

Roots $\quad x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$f(x) = \sin(x) + \frac{1}{x} + e^{x^2}$, Find roots? We need numerical methods

Roots of Equations

- Chapter 5
  Bracketing methods
  → we need interval to find roots
- Chapter 6
  Open methods
  → we only one value (initial guess) to Find roots

1- Bisection Method

2- False-Position Method

1- Simple fixed-point

2- Newton-Raphson

3- Secant and modified Secant

4- Multiple roots (NR)

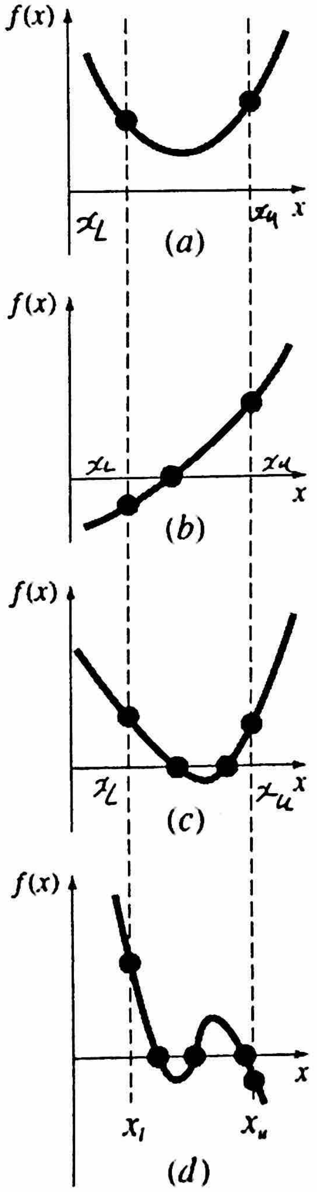
Chapter 5: Bracketing methods

* If a function $f(x)$ has a root (x_r) lies in the interval $[x_L, x_u]$ and $f(x_u) \cdot f(x_L) < 0$ (-ve), then the Bisection method computes x_r, as:

$$x_r = \frac{x_u + x_L}{2}$$

x_L = Lower

x_u = Upper



(a) $f(x_L) \cdot f(x_u) = (+)(+) = (+) \rightarrow$ No roots

(b) $f(x_L) \cdot f(x_u) = (-)(+) = (-) \rightarrow$ one root

(c) $f(x_L) \cdot f(x_u) = (+)(+)$ or $= (-)(-)$ $= (+)$ No roots or Even # of roots

(d) $f(x_L) \cdot f(x_u) = (+)(-) =$ / $(-)(+) =$ $(-) \rightarrow$ one root or odd # of roots

Solution procedure

① We have x_L and x_u, make sure $f(x_L) \cdot f(x_u) < 0$, If not, we cannot use Bisection method (No solution)

② Find $x_r = \dfrac{x_L + x_u}{2}$

③ Compute $f(x_r)$, and:

- If $f(x_L) \cdot f(x_r) = 0$, then x_r = exact root
- If $f(x_r) \cdot f(x_L) > 0$ (+ve), $x_L = x_r$ and back to step 2
- If $f(x_r) \cdot f(x_L) < 0$ (−ve), $x_u = x_r$ and back to step 2

This method needs Iterations

Example : $f(x) = e^{+x} - 2$, $x_L = 0$ and $x_u = 1$

- Find x_r for $\epsilon_a < |5\%|$

Solution

Iteration (1)

- check $f(x_L) \cdot f(x_u) < 0$ $\Rightarrow$ $f(x_L) = f(0) = 1 - 2 = -1$

$f(x_u) = f(1) = e^{1} - 2 = 0.73$

$\Rightarrow f(x_L) \cdot f(x_u) = (-)(+) = (-)$

- $x_r = \dfrac{x_L + x_u}{2} = \dfrac{0+1}{2} \Rightarrow x_r = 0.5$

- $f(x_r) = f(0.5) = e^{+0.5} - 2 = -0.35$

$f(x_L) \cdot f(x_r) = (-)(-) = + \Rightarrow x_L = x_r = 0.5$

Iteration 2

$x_L = 0.5$, $x_u = 1$ (no need to check $f(x_L) \cdot f(x_u) < 0$)

$x_r = \dfrac{x_L + x_u}{2} = \dfrac{0.5 + 1}{2} = 0.75$

$f(x_r) = f(0.75) = e^{+0.75} - 2 = 0.12$ (+)

$f(x_L) \cdot f(x_r) = f(0.5) \cdot f(0.75) = (-)(+) = (-)$

$x_u = x_r = 0.75$

Iteration 3

$x_L = 0.5 \qquad x_u = 0.75$

$$x_r = \frac{x_L + x_u}{2} = \frac{0.5 + 0.75}{2} = 0.625$$

$$f(x_r) = f(0.625) = e^{+0.625} - 2 = -0.13$$

$$f(x_L) \cdot f(x_r) = f(0.5) \cdot f(0.625) = (-)(-) = + \qquad x_L = x_r = 0.625$$

Till when do we need to find iterations?

Till we reach ε_a. In this example, $\varepsilon_a < 5\%$

Iteration	x_L	x_u	x_r	Sign $f(x_L)f(x_r)$	ε_a
1	0	1	0.5	+	N/A
2	0.5	1	0.75	−	$\varepsilon_a = \left\lvert \frac{0.75 - 0.5}{0.75} \right\rvert 100\% = 33.3\%$
3	0.5	0.75	0.625	+	$\varepsilon_a = \left\lvert \frac{0.625 - 0.75}{0.625} \right\rvert 100\% = 20\%$
4	0.625	0.75	0.6875	+	$\varepsilon_a = \left\lvert \frac{0.6875 - 0.625}{0.6875} \right\rvert 100\% = 9\%$
5	0.6875	0.75	0.71875	(−)	$\varepsilon_a = \left\lvert \frac{0.71875 - 0.6875}{0.71875} \right\rvert 100\% = 4.3\%$

$$\varepsilon_a = \left\lvert \frac{\text{Present value of } x_r - \text{Previous value of } x_r}{\text{Present value of } x_r} \right\rvert 100\%$$

$$\boxed{x_r = 0.71875}$$

Iteration 4

$x_l = 0.625 \quad , \quad x_u = 0.75$

$$x_r = \frac{x_L + x_u}{2} = \frac{0.625 + 0.75}{2} = 0.6875$$

$$f(x_r) = f(0.6875) = e^{0.6875} - 2 = -0.01$$

$$f(x_L) \cdot f(x_r) = (-)(-) = + \qquad x_L = x_r = 0.6875$$

Iteration 5

$x_L = 0.6875 \quad , \quad x_u = 0.75$

$$x_r = \frac{x_L + x_u}{2} = 0.71875$$

- **note**: approx error

In Bisection method, If given x_L, x_u and E_a we can find number of required iteration (n)

$$n = \log_2 \left(\frac{\Delta x^0}{E_a} \right) \qquad \Delta x^0 = x_u - x_L$$

Example $f(x) = x e^{-x} - 2$

$x_L = 0$, $x_u = 1$ and $E_a = 0.022$

Find required # of iteration to reach $E_a = 0.022$

Solution

$$n = \log_2 \left(\frac{\Delta x^0}{E_a} \right) = \log_2 \left(\frac{1-0}{0.022} \right) = \log_2 \left(\frac{1}{0.022} \right)$$

$n = 5.5 \Rightarrow n = 6$ iterations not 5

Example $f(x) = x e^{-x} - 2$ $\quad x_L = 0$, $x_u = 1$

Find E_a after $n=3$ iterations

$$n = \log_2 \left(\frac{\Delta x^0}{E_a} \right), \quad 2^{()} = 2^{()}$$

$$\Rightarrow E_a^{(n)} = \frac{\Delta x^0}{2^n}$$

$$E_a^{(3)} = \frac{1-0}{2^3} \Rightarrow \boxed{E_a = 0.125}$$

$$E_a^{(6)} = \frac{1}{2^6} \Rightarrow \boxed{E_a = 0.015625}$$