2.5

DIFFERENTIAL EQUATIONS OF MOTION OF A DEFORMABLE BODY

In previous sections, we determined the stress components needed to specify the state of stress at a *point* 0 in a deformed body for a given set of orthogonal coordinate axes (x, y, z). We derived transformation equations that define the state of stress at point 0 for any other set of orthogonal axes (X, Y, Z) rotated with respect to (x, y, z). We derived relations that give at point 0 the principal stresses and their directions, the maximum shear stress, the octahedral normal and shear stresses, and the hydrostatic and deviatoric states of stress.

In this section, we derive differential equations of motion of a deformable solid body (differential equations of equilibrium if the deformed body has zero acceleration). These equations are needed when the theory of elasticity is used to derive load-stress and load-deflection relations for a member. We consider a general deformed body and choose a differential volume element at point 0 in the body as indicated in Fig. 2.14. The form of the differential equations of motion depends on the type oforthogonal coordinate axes employed. We choose rectangular coordinate axes (x, y, z) whose directions are parallel to the edges of the volume element. *In this book, we restrict our consideration mainly to small displacements and, therefore, do not distinguish between coordinate axes in the deformed state and in the undeformed state* (Boresi and Chong, 1987). Six cutting planes bound the volume element shown as a free-body diagram in Fig. 2.15. In general, the state of stress changes with the location of point 0. In particular, the stress components undergo changes from one face of the volume element to another face. Body forces (B_x, B_y, B_z) are included in the free-body diagram.

To write the differential equations of motion, each stress component must be multiplied by the area on which it acts and each body force must be multiplied by the volume of the element since (B_x, B_y, B_z) have dimensions of force per unit volume. The equations of motion for the volume element in Fig. 2.15 are then obtained by summation of these forces and summation of moments. In Sec. 2.3 we have already used summation of moments to obtain the stress symmetry conditions [Eqs. (2.4)]. Summation of forces in the x direction gives*

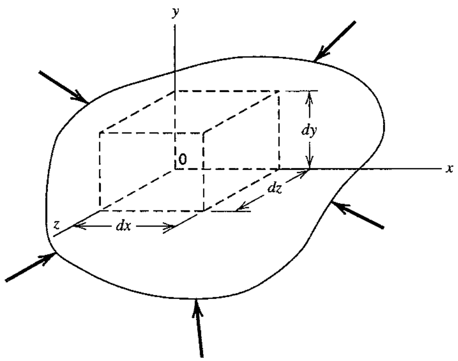

Figure 2.14 General deformed body.

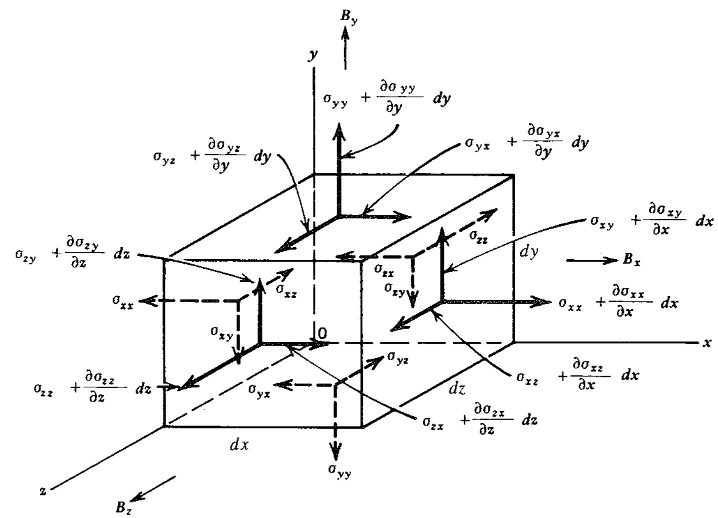

Figure 2.15 Stress components showing changes from face to face along with body force per unit volume including inertial forces.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x = 0$$

where σ_{xx}, $\sigma_{yx} = \sigma_{xy}$, and $\sigma_{xz} = \sigma_{zx}$ are stress components in the x direction and B_x is the body force per unit volume in the x direction including inertial (acceleration) forces. Summation of forces in the y and z directions yields similar results. The three equations of motion are thus

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0 \tag{2.45}$$

We use Eqs. (2.45) in the treatment of torsion of noncircular sections (Chapter 6).

As noted earlier, the form of the differential equations of motion depends on the coordinate axes; Eqs. (2.45) were derived for rectangular coordinate axes. In this book we need differential equations of motion in terms of cylindrical coordinates and plane polar coordinates. These are not derived here; instead, we present the

* Note σ_{xx} on the left face of the element goes to $\sigma_{xx} + d\sigma_{xx} = \sigma_{xx} + (\partial \sigma_{xx}/\partial x)\,dx$ on the right face of the element, with similar changes for the other stress components (Fig. 2.15).

most general form from the literature (Boresi and Chong, 1987, pp. 218–222) and show how the general form can be reduced to desired forms. The equations of motion relative to orthogonal curvilinear coordinates (x, y, z) (see Fig. 2.16), are

$$\frac{\partial(\beta\gamma\sigma_{xx})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yx})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{zx})}{\partial z} + \gamma\sigma_{yx}\frac{\partial\alpha}{\partial y} + \beta\sigma_{zx}\frac{\partial\alpha}{\partial z} - \gamma\sigma_{yy}\frac{\partial\beta}{\partial x} - \beta\sigma_{zz}\frac{\partial\gamma}{\partial x} + \alpha\beta\gamma B_x = 0$$

$$\frac{\partial(\beta\gamma\sigma_{xy})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yy})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{zy})}{\partial z} + \alpha\sigma_{zy}\frac{\partial\beta}{\partial z} + \gamma\sigma_{xy}\frac{\partial\beta}{\partial x} - \alpha\sigma_{zz}\frac{\partial\gamma}{\partial y} - \gamma\sigma_{xx}\frac{\partial\alpha}{\partial y} + \alpha\beta\gamma B_y = 0$$

$$\frac{\partial(\beta\gamma\sigma_{xz})}{\partial x} + \frac{\partial(\gamma\alpha\sigma_{yz})}{\partial y} + \frac{\partial(\alpha\beta\sigma_{zz})}{\partial z} + \beta\sigma_{xz}\frac{\partial\gamma}{\partial x} + \alpha\sigma_{yz}\frac{\partial\gamma}{\partial y} - \beta\sigma_{xx}\frac{\partial\alpha}{\partial z} - \alpha\sigma_{yy}\frac{\partial\beta}{\partial z} + \alpha\beta\gamma B_z = 0 \quad (2.46)$$

where (α, β, γ) are metric coefficients that are functions of the coordinates (x, y, z). They are defined by

$$ds^2 = \alpha^2\,dx^2 + \beta^2\,dy^2 + \gamma^2\,dz^2 \quad (2.47)$$

where ds is the differential arc length representing the diagonal of a volume element (Fig. 2.16) with edge lengths $\alpha\,dx$, $\beta\,dy$, and $\gamma\,dz$, and where (B_x, B_y, B_z) are the components of body force per unit volume including inertial forces. For rectangular coordinates, $\alpha = \beta = \gamma = 1$ and Eqs. (2.46) reduce to Eqs. (2.45).

Specialization of Equations (2.46)

Commonly employed orthogonal curvilinear systems in three-dimensional problems are the cylindrical coordinate system (r, θ, z) and spherical coordinate system (r, θ, ϕ); in plane problems, the plane polar coordinate system (r, θ) is frequently used. We will now specialize Eqs. (2.46) for these systems.

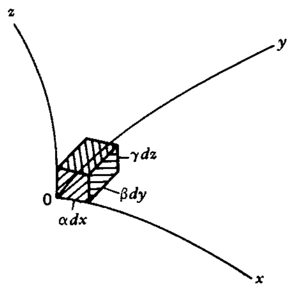

Figure 2.16 Orthogonal curvilinear coordinates.

(a) Cylindrical Coordinate System (r, θ, z). In Eqs. (2.46), we let $x = r$, $y = \theta$, $z = z$. Then the differential length ds is defined by the relation

$$ds^2 = dr^2 + r^2\,d\theta^2 + dz^2 \quad (2.48)$$

A comparison of Eqs. (2.47) and (2.48) yields

$$\alpha = 1, \qquad \beta = r, \qquad \gamma = 1 \quad (2.49)$$

Substituting Eq. (2.49) into Eqs. (2.46), we obtain the differential equations of motion

$$\begin{aligned}
&\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{\partial\sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + B_r = 0\\
&\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + B_\theta = 0\\
&\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + B_z = 0
\end{aligned} \quad (2.50)$$

where $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{rz}, \sigma_{\theta z})$ represent stress components defined relative to cylindrical coordinates (r, θ, z). We use Eqs. (2.50) in Chapter 11 to derive load-stress and load-deflection relations for thick-wall cylinders.

(b) Spherical Coordinate System (r, θ, ϕ). In Eqs. (2.46), we let $x = r$, $y = \theta$, $z = \phi$, where r is the radial coordinate, θ the colatitude, and ϕ the longitude. Since the differential length ds is defined by

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 \quad (2.51)$$

comparison of Eqs. (2.47) and (2.51) yields

$$\alpha = 1, \qquad \beta = r, \qquad \gamma = r\sin\theta \quad (2.52)$$

Substituting Eq. (2.52) into Eqs. (2.46), we obtain the differential equations of motion

$$\begin{aligned}
&\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi r}}{\partial\phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{\theta r}\cot\theta) + B_r = 0\\
&\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\theta}}{\partial\phi} + \frac{1}{r}[(\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + 3\sigma_{r\theta}] + B_\theta = 0\\
&\frac{\partial\sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{1}{r}(3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta) + B_\phi = 0
\end{aligned} \quad (2.53)$$

where $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \sigma_{r\theta}, \sigma_{r\phi}, \sigma_{\theta\phi})$ are defined relative to spherical coordinates (r, θ, ϕ).

(c) Plane Polar Coordinate System (r, θ). In plane-stress problems relative to (x, y) coordinates, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, and the remaining stress components are functions of (x, y) only (Sec. 2.4). Letting $x = r, y = \theta, z = z$ in Eqs. (2.50) and noting that $\sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = (\partial/\partial z) = 0$, we obtain from Eq. (2.50)

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + B_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\frac{\sigma_{r\theta}}{r} + B_\theta = 0 \quad (2.54)$$

2.6 DEFORMATION OF A DEFORMABLE BODY

In the first four sections of this chapter, we examined the six stress components that define the state of stress at a point in a loaded member, derived the transformation equations of stress, and derived expressions for the maximum principal stress, maximum shear stress, and maximum octahedral shear stress at a point. These relations are of interest throughout most of the book. Differential equations of equilibrium (differential equations of motion for members being accelerated) were derived in Sec. 2.5. These are needed in chapters in which the theory of elasticity is used to derive load-stress and load-deflection relations. Additionally, differential equations of compatibility, needed in the theory of elasticity, are derived in Sec. 2.8; the derivation employs small displacement approximations and the associated strain-displacement relations. Although small displacements are considered in most applications of this book, more general finite strain-displacement relations are derived in this chapter so that the reader may better understand the approximations that lead to the strain-displacement relations of small-displacement theory.

In the derivation of strain-displacement relations for a member, we consider the member first to be unloaded (undeformed and unstressed) and next to be loaded (stressed and deformed). We let R represent the closed region occupied by the undeformed member and R^* the closed region occupied by the deformed member. Asterisks are used to designate quantities associated with the deformed state of members throughout the book.

Let (x, y, z) be rectangular coordinates (Fig. 2.17). A particle P is located at the general coordinate point (x, y, z) in the undeformed body. Under a deformation, the particle moves to a point (x^*, y^*, z^*) in the deformed state defined by the equations

$$x^* = x^*(x, y, z)$$
$$y^* = y^*(x, y, z)$$
$$z^* = z^*(x, y, z) \quad (2.55)$$

where the values of (x, y, z) are restricted to region R and (x^*, y^*, z^*) are restricted to region R^*. Equations (2.55) define the final location of a particle P that lies at a given point (x, y, z) in the undeformed member. It is assumed that the functions (x^*, y^*, z^*) are continuous and differentiable in the independent variables (x, y, z), since a discontinuity of these functions would imply a rupture of the member. Mathematically, this means that Eqs. (2.55) may be solved for single-valued solu-

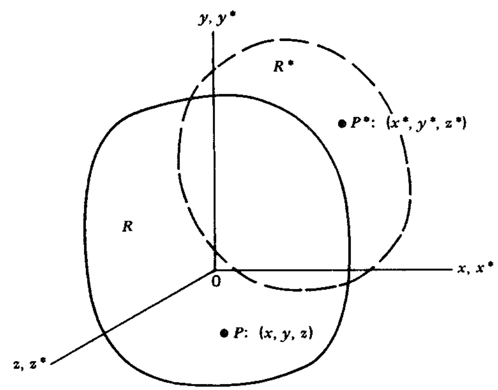

Figure 2.17 Location of general point P in undeformed and deformed body.

tions of (x, y, z); that is,

$$x = x(x^*, y^*, z^*)$$
$$y = y(x^*, y^*, z^*)$$
$$z = z(x^*, y^*, z^*) \quad (2.56)$$

Equations (2.56) define the initial location of a particle P that lies at point (x^*, y^*, z^*) in the deformed member. Functions (x, y, z) are continuous and differentiable in the independent variables (x^*, y^*, z^*).

When (x^*, y^*, z^*) are used as independent variables [Eq. (2.56)], the point of view is that of the *Eulerian or spatial coordinate method.* When (x, y, z) are used as independent variables, the point of view is that of the *Lagrangian or material coordinate method.* It may be shown that for classical, small-displacement theories of elasticity and plasticity, it is not necessary to distinguish between the variables (x^*, y^*, z^*) and (x, y, z). We employ material coordinates in this book.

2.7 STRAIN THEORY. TRANSFORMATION OF STRAIN. PRINCIPAL STRAINS*

The theory of stress of a continuous medium rests solely on Newton's laws. As will be shown in this section, the theory of strain rests solely on geometric concepts. Both the theories of stress and strain are, therefore, independent of material behavior and, as such, are applicable to the study of all materials. Furthermore, although the theories of stress and strain are based on different physical concepts,

* The theory presented in this article includes quadratic terms in the displacement components (u, v, w) and in the engineering strain ϵ_E. One may discard all quadratic terms in u, v, w, and ϵ_E and directly obtain the theory of strain for small deformations. (See Sec. 2.8.)

mathematically, they are equivalent, as will become evident in the following discussion.

Strain of a Line Element

When a body is deformed, the particle at point $P:(x, y, z)$ passes to the point $P^*:(x^*, y^*, z^*)$ (Fig. 2.18). Also, the particle at point $Q:(x + dx, y + dy, z + dz)$ passes to the point $Q^*:(x^* + dx^*, y^* + dy^*, z^* + dz^*)$, and the infinitesimal line element $PQ = ds$ passes into the line element $P^*Q^* = ds^*$. We define the *engineering strain* ϵ_E of the line element $PQ = ds$ as

$$\epsilon_E = \frac{ds^* - ds}{ds} \tag{2.57}$$

Therefore, by the definition, $\epsilon_E > -1$. Equation (2.57) is employed widely in engineering.

By Eqs. (2.55), we obtain the total differential

$$dx^* = \frac{\partial x^*}{\partial x}dx + \frac{\partial x^*}{\partial y}dy + \frac{\partial x^*}{\partial z}dz \tag{2.58}$$

with similar expressions for dy^*, dz^*. Noting that

$$\begin{aligned} x^* &= x + u \\ y^* &= y + v \\ z^* &= z + w \end{aligned} \tag{2.59}$$

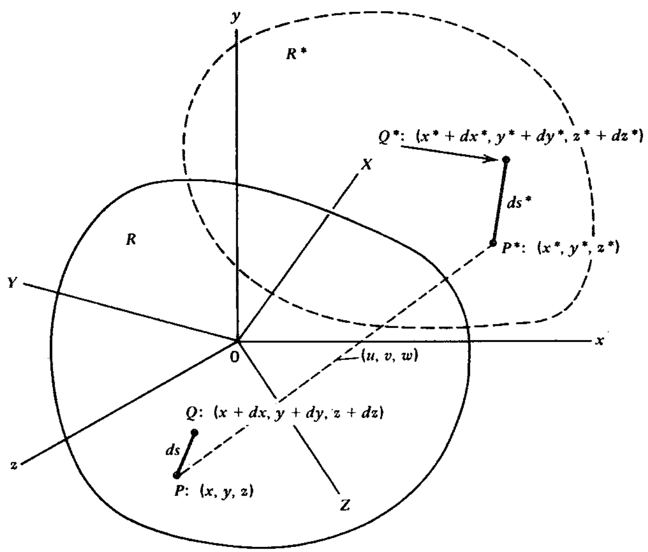

Figure 2.18 Line segment PQ in undeformed and deformed body.

where (u, v, w) denote the (x, y, z) components of the displacement of P to P^*, and also noting that

$$\begin{aligned} (ds)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ (ds^*)^2 &= (dx^*)^2 + (dy^*)^2 + (dz^*)^2 \end{aligned} \tag{2.60}$$

we find† [retaining quadratic terms in derivatives of (u, v, w)]

$$\begin{aligned} M = \frac{1}{2}\left[\left(\frac{ds^*}{ds}\right)^2 - 1\right] &= \epsilon_E + \tfrac{1}{2}\epsilon_E^2 = l^2\epsilon_{xx} + lm\epsilon_{xy} + ln\epsilon_{xz} \\ &+ ml\epsilon_{yx} + m^2\epsilon_{yy} + mn\epsilon_{yz} + nl\epsilon_{zx} + nm\epsilon_{zy} + n^2\epsilon_{zz} \\ &= l^2\epsilon_{xx} + m^2\epsilon_{yy} + n^2\epsilon_{zz} + 2lm\epsilon_{xy} + 2ln\epsilon_{xz} + 2mn\epsilon_{yz} \end{aligned} \tag{2.61}$$

where M is called the *magnification factor* and

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\ \epsilon_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \\ \epsilon_{xy} = \epsilon_{yx} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\ \epsilon_{xz} = \epsilon_{zx} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\ \epsilon_{yz} = \epsilon_{zy} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) \end{aligned} \tag{2.62}$$

are the finite strain-displacement relations‡ and where

$$l = \frac{dx}{ds}, \qquad m = \frac{dy}{ds}, \qquad n = \frac{dz}{ds} \tag{2.63}$$

are the direction cosines of line element ds.

We may interpret the quantities ϵ_{xx}, ϵ_{yy}, ϵ_{zz} physically, by considering line elements ds that lie parallel to the (x, y, z) axes, respectively. For example, let the line element ds (Fig. 2.18) lie parallel to the x axis. Then $l = 1$, $m = n = 0$, and

† Although one may compute ϵ_E directly from Eq. (2.57), it is mathematically simpler to form the quantity $M = \frac{1}{2}[(ds^*/ds)^2 - 1] = \frac{1}{2}[(1 + \epsilon_E)^2 - 1] = \epsilon_E + \frac{1}{2}\epsilon_E^2$. Then one may compute ϵ_E from Eq. (2.61). For small ϵ_E (Sec. 2.8), $\epsilon_E \cong M$. A more detailed derivation of Eq. (2.61) is given by Boresi and Chong (1987, Sec. 2-6).

‡ In small displacement theory, the quadratic terms in Eqs. (2.62) are neglected. Then, Eqs. (2.62) reduce to Eqs. (2.81).

Eq. (2.61) yields

$$M_x = \epsilon_{Ex} + \tfrac{1}{2}\epsilon_{Ex}^2 = \epsilon_{xx} \tag{2.61a}$$

where M_x and ϵ_{Ex} denote the magnification factor and the engineering strain of the element ds (parallel to the x direction). Hence, ϵ_{xx}, physically, is the magnification factor of the line element at P that lies initially in the x direction. In particular, if the engineering strain is small ($\epsilon_{Ex} \ll 1$), we obtain the result $\epsilon_{xx} \approx \epsilon_{Ex}$: namely, that ϵ_{xx} is approximately equal to the engineering strain for small strains. Similarly, for the cases where initially ds lies parallel to the y axis and then the z axis, we obtain

$$\begin{aligned} M_y &= \epsilon_{Ey} + \tfrac{1}{2}\epsilon_{Ey}^2 = \epsilon_{yy} \\ M_z &= \epsilon_{Ez} + \tfrac{1}{2}\epsilon_{Ez}^2 = \epsilon_{zz} \end{aligned} \tag{2.61b}$$

Thus, $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz})$ physically represent the magnification factors for line elements that initially lie parallel to the (x, y, z) axes, respectively.

To obtain a physical interpretation of the components ϵ_{xy}, ϵ_{xz}, ϵ_{yz}, it is necessary to determine the rotation between two line elements initially parallel to the (x, y) axes, (x, z) axes, and (y, z) axes, respectively. To do this, we first determine the final direction of a single line element under the deformation. Then, we use this result to determine the rotation between two line elements.

Final Direction of Line Element

As a result of the deformation, the line element ds: (dx, dy, dz) deforms into the line element ds^*: (dx^*, dy^*, dz^*). By definition, the direction cosines of ds and ds^* are

$$\begin{aligned} l &= \frac{dx}{ds}, \qquad m = \frac{dy}{ds}, \qquad n = \frac{dz}{ds} \\ l^* &= \frac{dx^*}{ds^*}, \qquad m^* = \frac{dy^*}{ds^*}, \qquad n^* = \frac{dz^*}{ds^*} \end{aligned} \tag{2.64}$$

Alternatively, we may write

$$l^* = \frac{dx^*}{ds}\frac{ds}{ds^*}, \qquad m^* = \frac{dy^*}{ds}\frac{ds}{ds^*}, \qquad n^* = \frac{dz^*}{ds}\frac{ds}{ds^*} \tag{2.65}$$

By Eqs. (2.58) and (2.59), we find

$$\begin{aligned} \frac{dx^*}{ds} &= \left(1 + \frac{\partial u}{\partial x}\right) l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n \\ \frac{dy^*}{ds} &= \frac{\partial v}{\partial x} l + \left(1 + \frac{\partial v}{\partial y}\right) m + \frac{\partial v}{\partial z} n \\ \frac{dz^*}{ds} &= \frac{\partial w}{\partial x} l + \frac{\partial w}{\partial y} m + \left(1 + \frac{\partial w}{\partial z}\right) n \end{aligned} \tag{2.66}$$

and by Eq. (2.57)

$$\frac{ds}{ds^*} = \frac{1}{1 + \epsilon_E} \tag{2.67}$$

Hence, Eqs. (2.65), (2.66), and (2.67) yield

$$\begin{aligned} (1 + \epsilon_E) l^* &= \left(1 + \frac{\partial u}{\partial x}\right) l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n \\ (1 + \epsilon_E) m^* &= \frac{\partial v}{\partial x} l + \left(1 + \frac{\partial v}{\partial y}\right) m + \frac{\partial v}{\partial z} n \\ (1 + \epsilon_E) n^* &= \frac{\partial w}{\partial x} l + \frac{\partial w}{\partial y} m + \left(1 + \frac{\partial w}{\partial z}\right) n \end{aligned} \tag{2.68}$$

Equations (2.68) represent the final direction cosines of line element ds when it passes into the line element ds^* under the deformation.

Rotation Between Two Line Elements (Definition of Shear Strain)

Next, let us consider two infinitesimal line elements PA and PB of lengths ds_1 and ds_2 emanating from point P. For simplicity, let PA be perpendicular to PB† (Fig. 2.19). Let the direction cosines of lines PA and PB be (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively. By the deformation, line elements PA, PB are transformed into line elements P^*A^*, P^*B^*, with direction cosines (l_1^*, m_1^*, n_1^*) and (l_2^*, m_2^*, n_2^*), respectively. Since PA is perpendicular to PB, by the definition of scalar product of vectors

$$\cos\frac{\pi}{2} = l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \tag{2.69}$$

Similarly, the angle θ^* between P^*A^* and P^*B^* is defined by

$$\cos\theta^* = l_1^* l_2^* + m_1^* m_2^* + n_1^* n_2^* \tag{2.70}$$

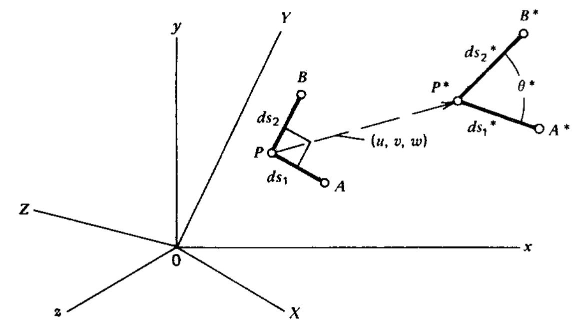

Figure 2.19 Line segments PA and PB before and after deformation.

† This restriction is not necessary but is used for simplicity. See Boresi and Chong (1987).

In turn, (l_1^*, m_1^*, n_1^*) and (l_2^*, m_2^*, n_2^*) are expressed in terms of (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively, by means of Eq. (2.68). Hence, by Eqs. (2.68), (2.69), and (2.70), we may write with Eqs. (2.62)

$$\begin{aligned}\gamma_{12} &= (1 + \epsilon_{E1})(1 + \epsilon_{E2}) \cos \theta^* \\ &= 2l_1 l_2 \epsilon_{xx} + 2m_1 m_2 \epsilon_{yy} + 2n_1 n_2 \epsilon_{zz} + 2(l_1 m_2 + l_2 m_1)\epsilon_{xy} \\ &\quad + 2(m_1 n_2 + m_2 n_1)\epsilon_{yz} + 2(l_1 n_2 + l_2 n_1)\epsilon_{xz}\end{aligned} \tag{2.71}$$

where γ_{12} is defined to be the *engineering shear strain* between line elements PA and PB as they are deformed into P^*A^* and P^*B^* (Fig. 2.19).

To obtain a physical interpretation of ϵ_{xy}, we now let PA and PB be oriented initially parallel to axes (x, y), respectively. Hence, $l_1 = 1$, $m_1 = n_1 = 0$ and $l_2 = n_2 = 0$, $m_2 = 1$. Then Eq. (2.71) yields the result

$$\gamma_{12} = \gamma_{xy} = 2\epsilon_{xy} \tag{2.72}$$

In other words, $2\epsilon_{xy}$ represents the engineering shear strain between two line elements initially parallel to the (x, y) axes, respectively. Similarly, we may consider PA and PB to be oriented initially parallel to the (y, z) axes and then to the (x, z) axes to obtain similar interpretations for ϵ_{yz}, ϵ_{xz}. Thus,

$$\gamma_{xy} = 2\epsilon_{xy}, \qquad \gamma_{yz} = 2\epsilon_{yz}, \qquad \gamma_{xz} = 2\epsilon_{xz} \tag{2.73}$$

represent the engineering shear strains between two line elements initially parallel to the (x, y), (y, z) and (x, z) axes, respectively.

If the strains ϵ_{E1}, ϵ_{E2} are small and the rotations are small (e.g., $\theta^* \approx \pi/2$), Eq. (2.71) yields the approximation

$$\gamma_{12} = (1 + \epsilon_{E1})(1 + \epsilon_{E2}) \cos \theta^* \approx \frac{\pi}{2} - \theta^* \tag{2.74}$$

and the engineering shear strain becomes approximately equal to the change in angle between line elements PA and PB.

Other results analogous to those of stress theory (Sec. 2.3 and 2.4) also hold. For example, the symmetric array

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} \tag{2.75}$$

is the *strain tensor*. Under a rotation of axes, the components of the strain tensor $(\epsilon_{xx}, \epsilon_{xy}, \epsilon_{xz}, \ldots)$ transform in exactly the same way as the stress tensor [Eqs. (2.15) and (2.17)]. [Compare Eqs. (2.5) and (2.75). Also compare Eqs. (2.11) and (2.61)]. To show this transformation, consider again axes (x, y, z) and (X, Y, Z), as in Sec. 2.4, Fig. 2.8 (also Fig. 2.18), and Table 2.2. The strain components ϵ_{XX}, ϵ_{XY}, $\epsilon_{XZ}, \ldots$, are defined with reference to axes (X, Y, Z) in the same manner as ϵ_{xx}, ϵ_{xy}, $\epsilon_{xz}, \ldots$, are defined relative to axes (x, y, z). Hence, ϵ_{XX} is the extensional strain of a line element at point P (Fig. 2.18) that lies in the direction of the X axis, and ϵ_{XY} and ϵ_{XZ} are shear components between line elements that are parallel to axes (X, Y) and (X, Z), respectively, and so on for ϵ_{YY}, ϵ_{ZZ}, ϵ_{YZ}. Hence, if we let element ds lie

parallel to the X axis, Eq. (2.61), with Table 2.2, yields

$$\epsilon_{XX} = l_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + n_1^2 \epsilon_{zz} + 2l_1 m_1 \epsilon_{xy} + 2l_1 n_1 \epsilon_{xz} + 2m_1 n_1 \epsilon_{yz} \tag{2.76a}$$

Similarly for the line elements that lie parallel to axes Y and Z, respectively, we have

$$\epsilon_{YY} = l_2^2 \epsilon_{xx} + m_2^2 \epsilon_{yy} + n_2^2 \epsilon_{zz} + 2l_2 m_2 \epsilon_{xy} + 2l_2 n_2 \epsilon_{xz} + 2m_2 n_2 \epsilon_{yz} \tag{2.76b}$$

$$\epsilon_{ZZ} = l_3^2 \epsilon_{xx} + m_3^2 \epsilon_{yy} + n_3^2 \epsilon_{zz} + 2l_3 m_3 \epsilon_{xy} + 2l_3 n_3 \epsilon_{xz} + 2m_3 n_3 \epsilon_{yz} \tag{2.76c}$$

Similarly, if we take line elements PA and PB parallel, respectively to axes X and Y (Fig. 2.19), Eqs. (2.71) and (2.73) yield the result

$$\begin{aligned}\tfrac{1}{2}\gamma_{XY} = \epsilon_{XY} &= l_1 l_2 \epsilon_{xx} + m_1 m_2 \epsilon_{yy} + n_1 n_2 \epsilon_{zz} + (l_1 m_2 + l_2 m_1)\epsilon_{xy} \\ &\quad + (m_1 n_2 + m_2 n_1)\epsilon_{yz} + (l_1 n_2 + l_2 n_1)\epsilon_{xz}\end{aligned} \tag{2.76d}$$

In a similar manner, we find

$$\begin{aligned}\tfrac{1}{2}\gamma_{YZ} = \epsilon_{YZ} &= l_2 l_3 \epsilon_{xx} + m_2 m_3 \epsilon_{yy} + n_2 n_3 \epsilon_{zz} + (l_2 m_3 + l_3 m_2)\epsilon_{xy} \\ &\quad + (m_2 n_3 + m_3 n_2)\epsilon_{yz} + (l_2 n_3 + l_3 n_2)\epsilon_{xz}\end{aligned} \tag{2.76e}$$

$$\begin{aligned}\tfrac{1}{2}\gamma_{XZ} = \epsilon_{XZ} &= l_1 l_3 \epsilon_{xx} + m_1 m_3 \epsilon_{yy} + n_1 n_3 \epsilon_{zz} + (l_1 m_3 + l_3 m_1)\epsilon_{xy} \\ &\quad + (m_1 n_3 + m_3 n_1)\epsilon_{yz} + (l_1 n_3 + l_3 n_1)\epsilon_{xz}\end{aligned} \tag{2.76f}$$

where (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the direction cosines of axes X, Y, and Z, respectively.

Equations (2.76) represent the transformation of the strain tensor $(\epsilon_{xx}, \epsilon_{yy}, \ldots, \epsilon_{yz})$ under a rotation from axes (x, y, z) to axes (X, Y, Z). (See Figs. 2.18 and 2.19 and also Fig. 2.8.)

Principal Strains

Under a deformation of a body (Sec. 2.6), any infinitesimal sphere in the body is deformed into an ellipsoid, called the *strain ellipsoid.* The principal axes of the strain ellipsoid have the directions of the principal axes of strain (see below) at the center of the ellipsoid in the deformed member. The radii of the infinitesimal sphere that pass into the principal axes of the strain ellipsoid are initially perpendicular to each other, and they coincide with the principal axes of strain in the undeformed body. Hence, through any point in an undeformed member, there exist three mutually perpendicular line elements that remain perpendicular under the deformation. The strains of these three line elements are called the *principal strains* at the point. We denote them by $(\epsilon_{E1}, \epsilon_{E2}, \epsilon_{E3})$ and the corresponding principal values of the magnification factor $M = \epsilon_E + \frac{1}{2}\epsilon_E^2$ are denoted by (M_1, M_2, M_3). By analogy with stress theory (Sec. 2.4), the principal values of the magnification factor are the three roots of the determinantal equation

$$\begin{vmatrix} \epsilon_{xx} - M & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} - M & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} - M \end{vmatrix} = 0 \tag{2.77a}$$

or

$$M^3 - \bar{I}_1 M^2 - \bar{I}_2 M - \bar{I}_3 = 0$$
$$M = \epsilon_E + \tfrac{1}{2}\epsilon_E^2 \tag{2.77b}$$

where

$$\bar{I}_1 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$$
$$\bar{I}_2 = -\begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{vmatrix} - \begin{vmatrix} \epsilon_{xx} & \epsilon_{xz} \\ \epsilon_{xz} & \epsilon_{zz} \end{vmatrix} - \begin{vmatrix} \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{yz} & \epsilon_{zz} \end{vmatrix}$$
$$= \epsilon_{xy}^2 + \epsilon_{xz}^2 + \epsilon_{yz}^2 - \epsilon_{xx}\epsilon_{yy} - \epsilon_{xx}\epsilon_{zz} - \epsilon_{yy}\epsilon_{zz}$$
$$\bar{I}_3 = \begin{vmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{vmatrix} \tag{2.78}$$

are the *strain invariants* [see Eqs. (2.19), (2.20), (2.21)]. Because of the symmetry of the determinant of Eq. (2.77a), the roots M_i:$i = 1, 2, 3$ are always real. Also since $\epsilon_{Ei} > -1$, $M_i > -1$.

The three principal strain directions associated with the three principal strains $(\epsilon_{E1}, \epsilon_{E2}, \epsilon_{E3})$, Eq. (2.77b), are obtained as the solution for (l, m, n) of the equations

$$l(\epsilon_{xx} - M) + m\epsilon_{xy} + n\epsilon_{xz} = 0$$
$$l\epsilon_{xy} + m(\epsilon_{yy} - M) + n\epsilon_{yz} = 0$$
$$l\epsilon_{xz} + m\epsilon_{yz} + n(\epsilon_{zz} - M) = 0$$
$$l^2 + m^2 + n^2 = 1 \tag{2.79}$$

Recall that only two of the first three of Eqs. (2.79) are independent. The solution $M = M_1$ yields the direction cosines for $\epsilon_E = \epsilon_{E1}$ and so on for $M = M_2(\epsilon_E = \epsilon_{E2})$, $M = M_3(\epsilon_E = \epsilon_{E3})$.

If (x, y, z) axes are principal strain axes, $\epsilon_{xx} = M_1$, $\epsilon_{yy} = M_2$, $\epsilon_{zz} = M_3$, $\epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0$ and the expressions for the strain invariants $\bar{I}_1, \bar{I}_2, \bar{I}_3$ to reduce to

$$\bar{I}_1 = M_1 + M_2 + M_3$$
$$\bar{I}_2 = -M_1M_2 - M_1M_3 - M_2M_3$$
$$\bar{I}_3 = M_1M_2M_3 \tag{2.80}$$

2.8 SMALL-DISPLACEMENT THEORY

The deformation theory developed in Sec. 2.6 and 2.7 is purely geometrical and the associated equations are exact. In the small-displacement theory, the quadratic terms in Eqs. (2.62) are discarded. Then

$$\epsilon_{xx} \cong \frac{\partial u}{\partial x}, \qquad \epsilon_{yy} \cong \frac{\partial v}{\partial y}, \qquad \epsilon_{zz} \cong \frac{\partial w}{\partial z}$$
$$\epsilon_{xy} \cong \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \qquad \epsilon_{xz} \cong \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \qquad \epsilon_{yz} \cong \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{2.81}$$

are the strain-displacement relations for small-displacement theory. Then, the magnification factor reduces to

$$M \cong \epsilon_E \tag{2.82}$$

The above approximations, which are the basis for small-displacement theory, imply that the strains and rotations (excluding rigid-body rotations) are small compared to unity. The latter condition is not necessarily satisfied in the deformation of thin flexible bodies, such as rods, plates, and shells. For these bodies the rotations may be large. Consequently, the small-displacement theory must be used with caution: It is usually applicable for massive (thick) bodies, but it may give results that are seriously in error when applied to thin flexible bodies.

Strain Compatibility Relations

The six strain components are determined by Eqs. (2.81) if the three displacement components (u, v, w) are known. However, the three displacement components (u, v, w) cannot be determined by the integration of Eqs. (2.81) if the six strain components are chosen arbitrarily. That is, certain relationships (the so-called *strain compatibility relations*) among the six strain components must exist in order that Eqs. (2.81) may be integrated to obtain the three displacement components. To illustrate this point, for simplicity, consider the case of *plane strain* relative to the (x, y) plane. This state of strain is defined by the condition that the displacement components (u, v) are functions of (x, y) only and w = constant. Then Eqs. (2.81) yield

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}, \qquad 2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0 \tag{a}$$

The strain compatibility condition is obtained by elimination of the two displacement components (u, v) from the three nonzero strain-displacement relations in Eqs. (a). This can be done by differentiation and addition as follows. Note that by differentiation, Eqs. (a) yield

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x\, \partial y^2}, \qquad \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2\, \partial y} \tag{b}$$

and

$$\frac{2\partial^2 \epsilon_{xy}}{\partial x\, \partial y} = \frac{\partial^3 u}{\partial x\, \partial y^2} + \frac{\partial^3 v}{\partial x^2\, \partial y} \tag{c}$$

Addition of the right-hand sides of Eqs. (b) shows that the right-hand side of Eq. (c) is obtained. Therefore, the relation

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{2\partial^2 \epsilon_{xy}}{\partial x\, \partial y} \tag{d}$$

among the three strain components exists. This result, valid for small strains, is known as the *strain compatibility relation for plane strain*. In the general case, a similar elimination of (u, v, w) from Eqs. (2.81) yields the results (Boresi and Chong, 1987; Sec. 2-16)

$$\begin{aligned}
&\frac{\partial^2 \epsilon_{yy}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial y^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x\,\partial y} \\
&\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \epsilon_{xz}}{\partial x\,\partial z} \\
&\frac{\partial^2 \epsilon_{zz}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial z^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y\,\partial z} \\
&\frac{\partial^2 \epsilon_{zz}}{\partial x\,\partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial z^2} = \frac{\partial^2 \epsilon_{yz}}{\partial z\,\partial x} + \frac{\partial^2 \epsilon_{zx}}{\partial y\,\partial z} \\
&\frac{\partial^2 \epsilon_{yy}}{\partial x\,\partial z} + \frac{\partial^2 \epsilon_{xz}}{\partial y^2} = \frac{\partial^2 \epsilon_{xy}}{\partial y\,\partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x\,\partial y} \\
&\frac{\partial^2 \epsilon_{xx}}{\partial y\,\partial z} + \frac{\partial^2 \epsilon_{yz}}{\partial x^2} = \frac{\partial^2 \epsilon_{xz}}{\partial x\,\partial y} + \frac{\partial^2 \epsilon_{xy}}{\partial x\,\partial z}
\end{aligned} \tag{2.83}$$

Equations (2.83) are known as the *strain compatibility equations of small-displacement theory*. It may be shown that if the strain components $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz})$ satisfy Eqs. (2.83), there exist displacement components (u, v, w) that are solutions of Eqs. (2.81). More fully, in the small-displacement theory, the functions $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz})$ are possible components of strain if, and only if, they satisfy Eqs. (2.83). For large displacement theory, the equivalent results are given by Murnahan (1951).

Strain-Displacement Relations for Orthogonal Curvilinear Coordinates

More generally, the strain-displacement relations [Eqs. (2.62)] may be written for orthogonal curvilinear coordinates (Fig. 2.16). The derivation of the expressions for $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{xz}, \epsilon_{yz})$ is a routine problem (Boresi and Chong, 1987). For small-displacement theory, the results are

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{\alpha}\left(\frac{\partial u}{\partial x} + \frac{v}{\beta}\frac{\partial \alpha}{\partial y} + \frac{w}{\gamma}\frac{\partial \alpha}{\partial z}\right) \\
\epsilon_{yy} &= \frac{1}{\beta}\left(\frac{\partial v}{\partial y} + \frac{w}{\gamma}\frac{\partial \beta}{\partial z} + \frac{u}{\alpha}\frac{\partial \beta}{\partial x}\right) \\
\epsilon_{zz} &= \frac{1}{\gamma}\left(\frac{\partial w}{\partial z} + \frac{u}{\alpha}\frac{\partial \gamma}{\partial x} + \frac{v}{\beta}\frac{\partial \gamma}{\partial y}\right) \\
\epsilon_{xy} &= \frac{1}{2}\left(\frac{1}{\beta}\frac{\partial u}{\partial y} + \frac{1}{\alpha}\frac{\partial v}{\partial x} - \frac{v}{\alpha\beta}\frac{\partial \beta}{\partial x} - \frac{u}{\alpha\beta}\frac{\partial \alpha}{\partial y}\right) \\
\epsilon_{xz} &= \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial w}{\partial x} + \frac{1}{\gamma}\frac{\partial u}{\partial z} - \frac{u}{\alpha\gamma}\frac{\partial \alpha}{\partial z} - \frac{w}{\alpha\gamma}\frac{\partial \gamma}{\partial x}\right) \\
\epsilon_{yz} &= \frac{1}{2}\left(\frac{1}{\beta}\frac{\partial w}{\partial y} + \frac{1}{\gamma}\frac{\partial v}{\partial z} - \frac{w}{\beta\gamma}\frac{\partial \gamma}{\partial y} - \frac{v}{\beta\gamma}\frac{\partial \beta}{\partial z}\right)
\end{aligned} \tag{2.84}$$

where (u, v, w) are the projections of the displacement vector of point (x, y, z) on the tangents to the respective coordinate lines at that point and (α, β, γ) are the metric coefficients of the coordinate system [Eq. (2.47)]. Equations (2.84) are easily specialized for particular coordinates. For cylindrical coordinates $x = r$, $y = \theta$, $z = z$ and then $\alpha = 1$, $\beta = r$, $\gamma = 1$; for spherical coordinates, $x = r$, $y = \theta$ = colatitude, $z = \phi$ = longitude and then $\alpha = 1$, $\beta = r$, $\gamma = r\sin\theta$ (see Sec. 2.5), etc.

Thus, we obtain for

Cylindrical Coordinates

$$\begin{aligned}
&\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z} \\
&\gamma_{r\theta} = 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{rz} = 2\epsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\
&\gamma_{\theta z} = 2\epsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}
\end{aligned} \tag{2.85}$$

Spherical Coordinates

$$\begin{aligned}
&\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \epsilon_{\phi\phi} = \frac{u}{r} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} \\
&\gamma_{r\theta} = 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{r\phi} = 2\epsilon_{r\phi} = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r} - \frac{w}{r} \\
&\gamma_{\theta\phi} = 2\epsilon_{\theta\phi} = \frac{1}{r}\left(\frac{\partial w}{\partial \theta} - w\cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}
\end{aligned} \tag{2.86}$$

Polar Coordinates

$$\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \gamma_{r\theta} = 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{2.87}$$

EXAMPLE 2.4
Three-Dimensional State of Strain

The parallelopiped in Fig. E2.4 is deformed into the shape indicated by the dashed straight lines (small displacements). The displacements are given by the following relations: $u = C_1xyz$, $v = C_2xyz$, and $w = C_3xyz$. (a) Determine the state of strain at point E when the coordinates of point E^* for the deformed body are (1.504, 1.002, 1.996). (b) Determine the normal strain at E in the direction of line EA. (c) Determine the shear strain at E for the undeformed orthogonal lines EA and EF.

SOLUTION

The magnitudes of C_1, C_2, and C_3 are obtained from the fact that the displacements of point E are known as follows: $u_E = 0.004$ m, $v_E = 0.002$ m, and

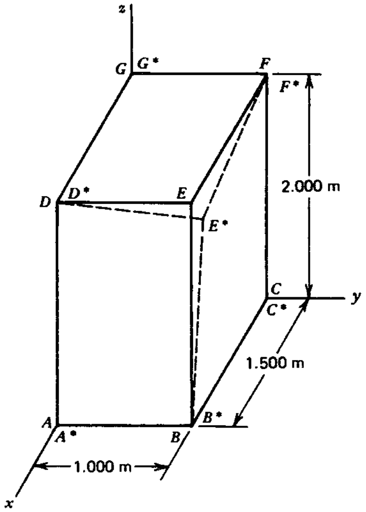

Figure E2.4

$w_E = -0.004$ m. Thus,

$$u = \frac{0.004}{3}xyz$$

$$v = \frac{0.002}{3}xyz$$

$$w = -\frac{0.004}{3}xyz$$

(a) The strain components for the state of strain at point E are given by Eqs. (2.81). At point E,

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{0.004}{3}yz = 0.00267, \qquad \epsilon_{yy} = 0.00200, \qquad \epsilon_{zz} = -0.00200$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{0.002}{3}yz + \frac{0.004}{3}xz\right) = 0.00267$$

$$\gamma_{xy} = 2\epsilon_{xy} = 0.00583, \qquad \gamma_{xz} = 2\epsilon_{xz} = -0.00007$$

$$\gamma_{yz} = 2\epsilon_{yz} = -0.00300$$

(b) Let the X axis lie along the line from E to A. The direction cosines of EA are $l_1 = 0$, $m_1 = -1/\sqrt{5}$, and $n_1 = -2/\sqrt{5}$. Equations (2.61) and (2.82) give the magnitude for ϵ_{XX}. Thus,

$$\epsilon_{XX} = \epsilon_{yy}m_1^2 + \epsilon_{zz}n_1^2 + 2\epsilon_{yz}m_1n_1$$

$$= \frac{0.00200}{5} - \frac{0.00200(4)}{5} - \frac{0.00300(2)}{5} = -0.00240$$

(c) Let the Y axis lie along the line from E to F. The direction cosines of EF are $l_2 = -1$, $m_2 = 0$, and $n_2 = 0$. The shear strain $\gamma_{XY} = 2\epsilon_{XY}$ is given by Eq. (2.76d). Thus,

$$\gamma_{XY} = 2\epsilon_{XY} = 2\epsilon_{xy}l_2m_1 + 2\epsilon_{xz}l_2n_1$$

$$= \frac{(0.00533)}{\sqrt{5}} + \frac{(-0.00007)(2)}{\sqrt{5}} = 0.00232$$

EXAMPLE 2.5
State of Strain in Torsion-Tension Member

A straight torsion-tension member with a solid circular cross section has a length $L = 6$ m and radius $R = 10$ mm. The member is subjected to tension and torsion loads that produce an elongation $\Delta L = 10$ mm and a rotation of one end of the member with respect to the other end of $\pi/3$ rad. Let the origin of the (r, θ, z) cylindrical coordinate axes lie at the centroid of one end of the member, with the z axis extending along the centroidal axis of the member. The deformations of the member are assumed to occur under conditions of constant volume. The end $z = 0$ is constrained so that only radial displacements are possible there. (a) Determine the displacements for any point in the member and the state of strain for a point on the outer surface. (b) Determine the principal strains for the point where the state of strain was determined.

SOLUTION

The change in radius ΔR for the member is obtained from the condition of constant volume. Thus,

$$\pi R^2 L = \pi(R + \Delta R)^2(L + \Delta L)$$

$$10^2(6 \times 10^3) = (10 + \Delta R)^2(6010)$$

$$\Delta R = -0.00832 \text{ mm}$$

(a) The displacements components

$$u = -0.000832r \text{ (mm)}$$

$$v = 0.0001745rz \text{ (mm)}$$

$$w = 0.001667z \text{ (mm)}$$

satisfy the displacement boundary conditions at $z = 0$. The strain components at the outer radius are given by Eqs. (2.85). They are (rounded to six decimal places)

$$\epsilon_{rr} = \frac{\partial u}{\partial r} = -0.000832, \qquad \epsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = -0.000832$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = 0.001667, \qquad \gamma_{r\theta} = 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0$$

$$\gamma_{rz} = 2\epsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0, \qquad \gamma_{\theta z} = 2\epsilon_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} = 0.001745$$

(b) The three principal strains are the three roots of a cubic equation, Eq. (2.77b), where the three invariants of strain are defined by Eqs. (2.78). Choose the (x, y, z) coordinate axes at the point on the outer surface of the member where the strain components have been determined in part (a). Let $x = r$, $y = \theta$, and $z = z$. From Eqs. (2.78),

$$\bar{I}_1 = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = -0.000832 - 0.000832 + 0.001667 \simeq 0$$

$$\begin{aligned}\bar{I}_2 &= -\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{rr}\epsilon_{zz} - \epsilon_{\theta\theta}\epsilon_{zz} + \epsilon_{r\theta}^2 + \epsilon_{rz}^2 + \epsilon_{\theta z}^2 \\ &= -(-0.000832)(-0.000832) - (-0.000832)(0.001667) \\ &\quad -(-0.000832)(0.001667) + \left(\frac{0.001745}{2}\right)^2 \\ &= +2.838 \times 10^{-6}\end{aligned}$$

$$\bar{I}_3 = \begin{vmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \epsilon_{zr} & \epsilon_{z\theta} & \epsilon_{zz} \end{vmatrix} = \begin{vmatrix} -0.000832 & 0 & 0 \\ 0 & -0.000832 & \dfrac{0.001745}{2} \\ 0 & \dfrac{0.001745}{2} & 0.001667 \end{vmatrix}$$

$$= 1.785 \times 10^{-9}$$

Substitution of these results into Eq. (2.77b) gives the following cubic equation in $\epsilon(= M)$:

$$\epsilon^3 - 2.838 \times 10^{-6}\epsilon - 1.785 \times 10^{-9} = 0$$

One principal strain, $\epsilon_{rr} = -0.000832$, is known. Factoring out this root, we find

$$\epsilon^2 - 0.000832\epsilon - 2.146 \times 10^{-6} = 0$$

Solution of this quadratic equation yields the remaining two principal strains. Thus, the three principal strains are

$$\begin{aligned}\epsilon_1 &= 0.001939 \\ \epsilon_2 &= -0.000832 \\ \epsilon_3 &= -0.001107\end{aligned}$$

EXAMPLE 2.6
Mohr's Circle for Plane Strain

A state of plane strain at a point in a body is given, with respect to the (x, y) axes, as $\epsilon_{xx} = 0.00044$, $\epsilon_{yy} = 0.00016$, and $\epsilon_{xy} = -0.00008$. Determine the principal strains in the (x, y) plane, the orientation of the principal axes of strain, the maximum shear strain, and the strain state on a block rotated by an angle of $\theta' = 25°$ measured counterclockwise with respect to the reference axes.

SOLUTION

Since the components of strain form a symmetric second-order tensor, they are transformed in precisely the same way as stresses. Thus, plane strain states can be represented by Mohr's circle in the same way as plane stress states. By analogy to the development for plane stress, Mohr's circle for plane strain is defined by the equation, [see Eq. (2.32)]

$$[\epsilon_{XX} - \tfrac{1}{2}(\epsilon_{xx} + \epsilon_{yy})]^2 + (\epsilon_{XY} - 0)^2 = \tfrac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2 \tag{a}$$

Equation (a) is the equation of a circle in the $(\epsilon_{XX}, \epsilon_{XY})$ plane with center coordinates

$$[\tfrac{1}{2}(\epsilon_{xx} + \epsilon_{yy}), 0] \tag{b}$$

and radius

$$R = \sqrt{\tfrac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2} \tag{c}$$

The orientation of the principal axes of strain is given by the angle θ, where

$$\tan 2\theta = \frac{2\epsilon_{xy}}{\epsilon_{xx} - \epsilon_{yy}} \tag{d}$$

and θ is measured with respect to the reference x axis, positive in the counterclockwise sense. The principal strains are

$$\epsilon_1 = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} + \sqrt{\tfrac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2} \tag{e}$$

$$\epsilon_2 = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} - \sqrt{\tfrac{1}{4}(\epsilon_{xx} - \epsilon_{yy})^2 + \epsilon_{xy}^2} \tag{f}$$

The maximum shear strain is simply the radius of the circle as given by Eq. (c).

For the data given, the state of strain may be expressed as $\epsilon_{xx} = 440\,\mu$, $\epsilon_{yy} = 160\,\mu$, and $\epsilon_{xy} = -80\,\mu$, where $\mu = 10^{-6}$. This representation of strain is known as *microstrain*. The Mohr's circle for this data is shown in Fig. E2.6*a*. By Eq. (b) and (c), the center of the circle is located at point C with coordinates $(300\,\mu, 0)$ and its radius is $R = 161\,\mu$. By Eqs. (e) and (f), the principal strains are

$$\epsilon_1 = 300\,\mu + 161\,\mu = 461\,\mu \tag{g}$$

$$\epsilon_2 = 300\,\mu - 161\,\mu = 139\,\mu \tag{h}$$

On Mohr's circle, they correspond to points Q and Q', respectively. The reference strain state is plotted at points $P(\epsilon_{xx}, \epsilon_{xy})$ and $P'(\epsilon_{yy}, -\epsilon_{xy})$. Note that the positive ϵ_{XY} axis is directed downward, as is done with the plane stress case. By Eq. (d) the principal axis corresponding to ϵ_1 in the body is located at an angle of $\theta = -14.87°$ with respect to the x axis. On Mohr's circle, this corresponds to an angle $2\theta = -29.74°$ from line CP to line CQ.

The maximum value of shear strain is $\epsilon_{XY(\max)} = R = 161\ \mu$. It occurs at an orientation of $\pm 45°$ from the principal axis for ϵ_1 ($\pm 90°$ from line CQ on Mohr's circle). Note that this is the maximum shear strain using the tensor definition of strain. The maximum engineering shear strain is $\gamma_{XY(\max)} = 2\epsilon_{XY(\max)} = 322\ \mu$ [Eq. (2.73)]. The strain state on a block at $\theta' = 25°$ (50° on Mohr's circle) is identified by points $S(\epsilon'_{xx}, \epsilon'_{xy})$ and $S'(\epsilon'_{yy}, -\epsilon'_{xy})$. By geometry of the circle, the strain

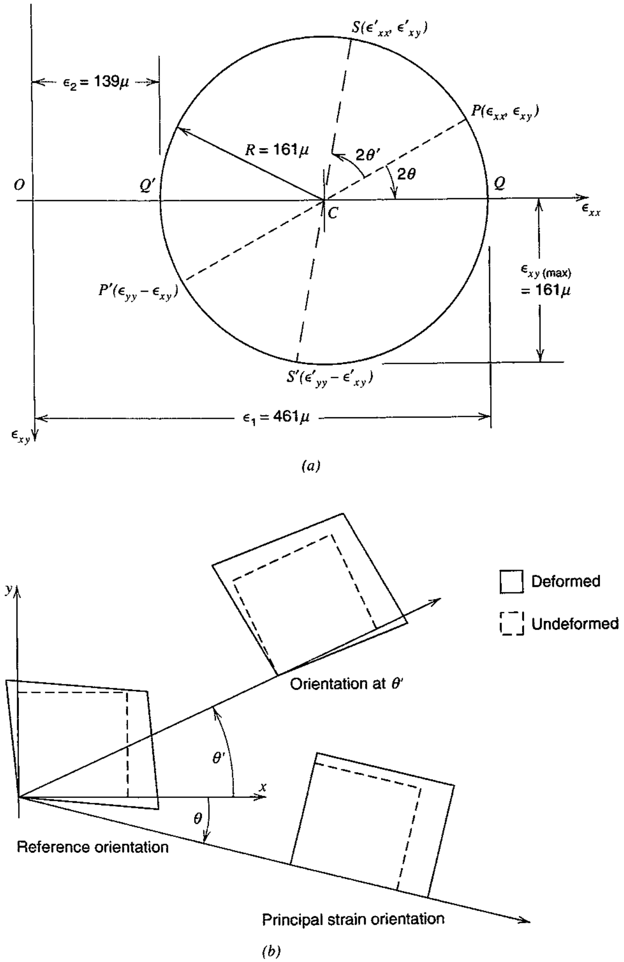

Figure E2.6 (*a*) Mohr's circle for plane strain. (*b*) Deformed element in three different orientations.

quantities are

$$\epsilon'_{xx} = OC + R\cos(2\theta' - 2\theta) = 329\,\mu$$
$$\epsilon'_{yy} = OC - R\cos(2\theta' - 2\theta) = 271\,\mu$$
$$\epsilon'_{xy} = -R\sin(2\theta' - 2\theta) = -295\,\mu$$

In Fig. E2.6b, the deformed shape of an element in the reference orientation is shown. Also illustrated is the deformed shape in the principal orientation, that is at an angle of $\theta = -14.87°$ with respect to the x axis. Notice that in this orientation, the deformed element is not distorted, since the shear strain is zero. Finally, the deformed shape at $\theta' = 25°$ with respect to the reference orientation is shown.

2.9 STRAIN MEASUREMENT. STRAIN ROSETTES

For members of complex shape subjected to loads, it may be mathematically impractical or impossible to derive analytical load-stress relations. Then, either numerical or experimental methods are used to obtain approximate results. Numerical methods (finite element methods) are treated in Chapter 19. Several experimental methods are used, the most common one being the use of strain gages. Strain gages are used to measure extensional strains on the free surface of a member or the axial extension/contraction of a bar. They cannot be used to measure the strain at an interior point of a member. To measure interior strains (or stresses), other techniques such as photoelasticity may be used, although this method has been largely superseded by modern numerical techniques. Nevertheless, photoelastic methods are still useful when augmented with modern computer data-acquisition techniques (Kobayashi, 1987). Additional experimental procedures are also available. They include holographic, Moiré, and laser speckle interferometry techniques. These specialized methods lie outside of the scope of this text (see Kobayashi, 1987). We shall discuss only the use of electrical resistance (bonded) strain gages. Electric strain gages are used to obtain average extensional strain over a given gage length. These gages are made of very fine wire or metal foil and are glued to the surface of the member being tested. When forces are applied to the member, the gage elongates or contracts with the member. The change in length of the gage alters its electrical resistance. The change in resistance can be measured and calibrated to indicate the average extensional strain that occurs over the gage length. To meet various requirements, gages are made in a variety of gage lengths, varying from 4 to 150 mm (approximately 0.15 to 6 in), and are designed for different environmental conditions.

A minimum of three extensional strain measurements in three different directions at a point on the surface of a member is required to determine the average state of strain at that point. Consequently, it is customary to cluster together three gages to form a *strain rosette* that may be cemented to the free surface of a member. Two common forms of rosettes are the *delta rosette* (with three gages spaced at 60° angles) and the *rectangular rosette* (with three gages spaced at 45° angles), Fig. 2.20. From the measurement of extensional strains along the gage arm directions (directions *a*, *b*, *c* in Fig. 2.20), one can determine the strain components $(\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy})$ at the point, relative to the (x, y) axes. Usually, one of the axes is

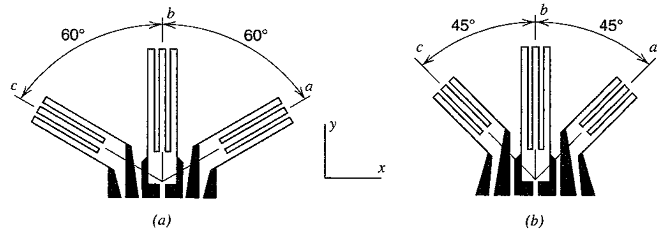

Figure 2.20 Rosette strain gages. (*a*) Delta rosette. (*b*) Rectangular rosette.

taken to be aligned with one arm of the rosette, say, the arm a. Hence, $\epsilon_{xx} = \epsilon_a$, the average extensional strain in the direction a. Then, the components $(\epsilon_{yy}, \epsilon_{xy})$ may be expressed in terms of the measured extensional strains ϵ_a, ϵ_b, and ϵ_c in the directions of the three rosette arms a, b, c, respectively. (See Example 2.7.)

EXAMPLE 2.7
Measurement of Strain on a Surface of a Member

A strain rosette with gages spaced at an angle θ is cemented to the free surface of a member (Fig. E2.7). Under a deformation of the member, the extensional strains measured by gages a, b, c are $\epsilon_a, \epsilon_b, \epsilon_c$, respectively. (a) Derive equations that determine the strain components $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}$ in terms of $\epsilon_a, \epsilon_b, \epsilon_c$ and θ. (b) Specialize the results for the delta rosette ($\theta = 60°$) and rectangular rosette ($\theta = 45°$).

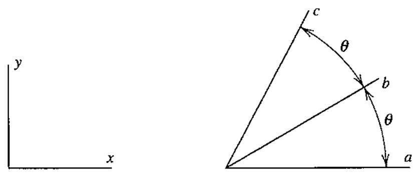

Figure E2.7

SOLUTION

(a) The direction cosines of arms a, b, c, are respectively,

$$(l_a, m_a, n_a) = (1, 0, 0), (l_b, m_b, n_b) = (\cos\theta, \sin\theta, 0), (l_c, m_c, n_c) = (\cos 2\theta, \sin 2\theta, 0)$$

The extensional strain of a line element in the direction (l, m, n) is given by Eq. (2.61). Hence, by Eq. (2.61), the extensional strains in the directions of arms a, b, c are

$$\begin{aligned} \epsilon_a &= \epsilon_{xx} \\ \epsilon_b &= \epsilon_{xx}(\cos^2\theta) + \epsilon_{yy}(\sin^2\theta) + 2\epsilon_{xy}(\cos\theta)(\sin\theta) \\ \epsilon_c &= \epsilon_{xx}(\cos^2 2\theta) + \epsilon_{yy}(\sin^2 2\theta) + 2\epsilon_{xy}(\cos 2\theta)(\sin 2\theta) \end{aligned} \tag{a}$$

Equations (a) are three equations that may be solved for ϵ_{xx}, ϵ_{yy}, and ϵ_{xy} in terms of ϵ_a, ϵ_b, and ϵ_c for a given angle θ. The solution is

$$\begin{aligned} \epsilon_{xx} &= \epsilon_a \\ \epsilon_{yy} &= \frac{(\epsilon_a - 2\epsilon_b)\sin 4\theta + 2\epsilon_c \sin 2\theta}{4\sin^2\theta \sin 2\theta} \\ \epsilon_{xy} &= \frac{2\epsilon_a(\sin^2\theta\cos^2 2\theta - \sin^2 2\theta\cos^2\theta) + 2(\epsilon_b \sin^2 2\theta - \epsilon_c \sin^2\theta)}{4\sin^2\theta\sin 2\theta} \end{aligned} \tag{b}$$

(b) For $\theta = 60°$, $\cos\theta = 1/2$, $\sin\theta = \sqrt{3}/2$, $\cos 2\theta = -1/2$, and $\sin 2\theta = \sqrt{3}/2$. Therefore, for $\theta = 60°$, Eqs. (b) yield

$$\begin{aligned} \epsilon_{xx} &= \epsilon_a \\ \epsilon_{yy} &= \frac{2(\epsilon_b + \epsilon_c) - \epsilon_a}{3} \\ \epsilon_{xy} &= \frac{\epsilon_b - \epsilon_c}{\sqrt{3}} \end{aligned} \tag{c}$$

For $\theta = 45°$, $\cos\theta = 1/\sqrt{2}$, $\sin\theta = 1/\sqrt{2}$, $\cos 2\theta = 0$, and $\sin 2\theta = 1$. Therefore, for $\theta = 45°$, Eqs. (b) yield

$$\epsilon_{xx} = \epsilon_a, \epsilon_{yy} = \epsilon_c, \epsilon_{xy} = \epsilon_b - \tfrac{1}{2}(\epsilon_a + \epsilon_c) \tag{d}$$

PROBLEMS
Sections 2.1–2.4

2.1. Let the state of stress at a point be specified by the following stress components: $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$, $\sigma_{xy} = -75$ MPa, $\sigma_{yz} = 65$ MPa, and $\sigma_{zx} = -55$ MPa. Determine the principal stresses, direction cosines for the three principal stress directions, and maximum shear stress.

2.2. Consider a state of stress in which the nonzero stress components are σ_{xx}, σ_{yy}, σ_{zz}, and σ_{xy}. Note that this is not a state of plane stress since $\sigma_{zz} \neq 0$. Consider another set of coordinate axes (X, Y, Z), with the Z axis coinciding with the z axis and the X axis located counterclockwise through angle θ from the x axis. Show that the transformation equations for this state of stress are identical to Eq. (2.30) or (2.31) for plane stress.

2.3. Let the state of stress at a point be specified by the following stress components: $\sigma_{xx} = 110$ MPa, $\sigma_{yy} = -86$ MPa, $\sigma_{zz} = 55$ MPa, $\sigma_{xy} = 60$ MPa, and $\sigma_{yz} = \sigma_{zx} = 0$. Determine the principal stresses, direction cosines of the principal stress directions, and maximum shear stress.

Ans. $\sigma_1 = 126.9$ MPa, $\sigma_2 = -102.9$ MPa, $\sigma_3 = 55.0$ MPa
$l_1 = 0.9625$, $m_1 = 0.2717$, $n_1 = 0$
$l_2 = 0.2717$, $m_2 = -0.9625$, $n_2 = 0$
$l_3 = 0$, $m_3 = 0$, $n_3 = -1$
$\sigma_{NS(\max)} = 114.9$ MPa

2.4. Solve Problem 2.3 using the results of Problem 2.2

2.5. Let the state of plane stress be specified by the following stress components: $\sigma_{xx} = 90$ MPa, $\sigma_{yy} = -10$ MPa, $\sigma_{xy} = 40$ MPa. Let the X axis lie in the (x, y) plane and be located at $\theta = \pi/6$ clockwise from the x axis. The direction cosines for the X axis are $l = \cos(-\pi/6) = 0.8660$, $m = \sin(-\pi/6) = -0.5000$, $n = 0$. Determine the normal and shear stresses on a plane perpendicular to the X axis; use Eqs. (2.10), (2.11), and (2.12).

Ans. $\sigma_{XX} = 30.36$ MPa, $\sigma_{XY} = 63.30$ MPa

In Problems 2.6 through 2.9, the Z axis for the transformed axes coincides with the z axis for the volume element on which the known stress components act.

2.6. The nonzero stress components are $\sigma_{xx} = 200$ MPa, $\sigma_{yy} = 100$ MPa, and $\sigma_{xy} = -50$ MPa. Determine the principal stresses and maximum shear stress. Determine the angle between the X axis and the x axis when the X axis is in the direction of the principal stress with largest absolute magnitude.

2.7. The nonzero stress components are $\sigma_{xx} = -90$ MPa, $\sigma_{yy} = 50$ MPa, and $\sigma_{xy} = 60$ MPa. Determine the principal stresses and maximum shear stress. Determine the angle between the X axis and the x axis when the X axis is in the direction of the principal stress with largest absolute magnitude.

Ans. $\sigma_1 = 72.2$ MPa, $\sigma_2 = -112.2$ MPa, $\sigma_3 = 0$, $\tau_{max} = 92.2$ MPa. X axis located 0.3543 rad clockwise from x axis.

2.8. The nonzero stress components are $\sigma_{xx} = 80$ MPa, $\sigma_{zz} = -60$ MPa, and $\sigma_{xy} = 30$ MPa. Determine the principal stresses and maximum shear stress. Determine the angle between the X axis and the x axis when the X axis is in the direction of the principal stress with largest absolute magnitude.

2.9. The nonzero stress components are $\sigma_{xx} = 150$ MPa, $\sigma_{yy} = 70$ MPa, $\sigma_{zz} = -80$ MPa, and $\sigma_{xy} = -45$ MPa. Determine the principal stresses and maximum shear stress. Determine the angle between the X axis and the x axis when the X axis is in the direction of the principal stress with largest absolute magnitude.

Ans. $\sigma_1 = 170.2$ MPa, $\sigma_2 = 49.8$ MPa, $\sigma_3 = -80$ MPa, $\tau_{max} = 125.1$ MPa. X axis located 0.4221 rad clockwise from the x axis.

2.10. Using transformation equations of plane stress, determine σ_{XX} and σ_{XY} for the X axis located 0.5000 rad clockwise from the x axis. The nonzero stress components are given in Problem 2.6.

2.11. Using transformation equations of plane stress, determine σ_{XX} and σ_{XY} for the X axis located 0.1500 rad counterclockwise from the x axis. The nonzero stress components are given in Problem 2.7.

Ans. $\sigma_{XX} = -69.1$ MPa, $\sigma_{XY} = 78.0$ MPa

2.12. Using transformation equations of stress (see Problem 2.2), determine σ_{XX} and σ_{XY} for the X axis located 1.00 rad clockwise from the x axis. The nonzero stress components are given in Problem 2.8.

2.13. Using transformation equations of stress (see Problem 2.2), determine σ_{XX} and σ_{XY} for the X axis located 0.70 rad counterclockwise from the x axis. The nonzero stress components are given in Problem 2.9.

Ans. $\sigma_{XX} = 72.5$ MPa, $\sigma_{XY} = -47.1$ MPa

2.14. Solve Problem 2.10 using Mohr's circle of stress.

2.15. Solve Problem 2.11 using Mohr's circle of stress.

2.16. Solve Problem 2.12 using Mohr's circle of stress.

2.17. Solve Problem 2.13 using Mohr's circle of stress.

2.18. A volume element at the free surface is shown in Fig. P2.18. The state of stress is plane stress with $\sigma_{xx} = 100$ MPa. Determine the other stress components.

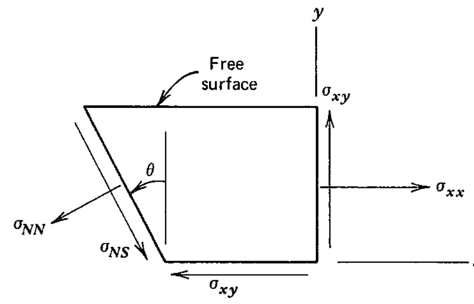

Figure P2.18

2.19. Determine the unknown stress components for the volume element in Fig. P2.19.

Ans. $\sigma_{xx} = 26.67$ MPa, $\sigma_{yy} = 172.50$ MPa

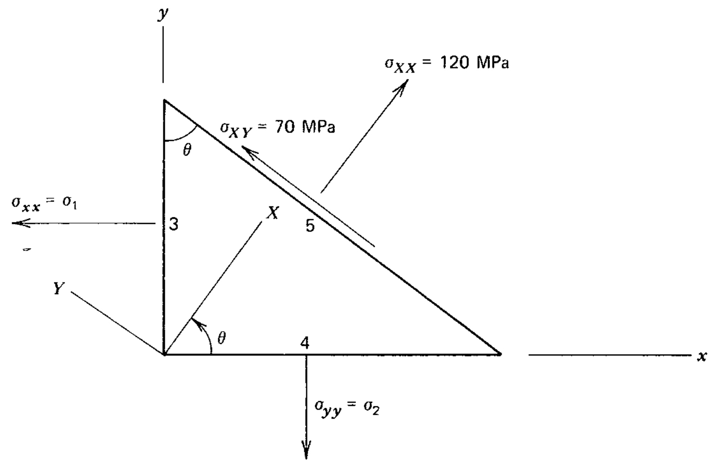

Figure P2.19

2.20. Determine the unknown stress components for the volume element in Fig. P2.20.

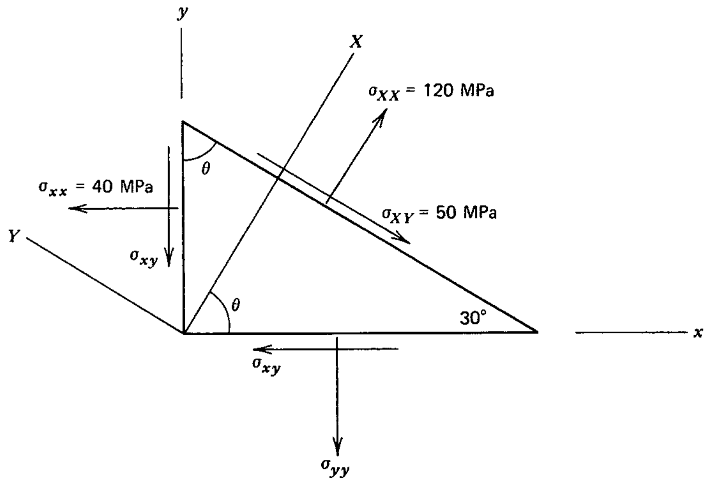

Figure P2.20

2.21. Determine the unknown stress components for the volume element in Fig. P2.21.

Ans. $\sigma_{xx} = -109.18$ MPa, $\sigma_{xy} = -10.01$ MPa

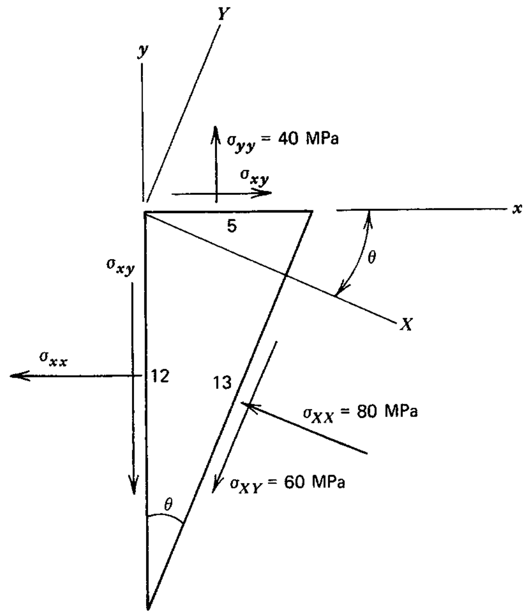

Figure P2.21

In Problems 2.22 through 2.26, determine the principal stresses, maximum shear stress, and octahedral shear stress.

2.22. The nonzero stress components are $\sigma_{xx} = -100$ MPa, $\sigma_{yy} = 60$ MPa, and $\sigma_{xy} = -50$ MPa.

2.23. The nonzero stress components are $\sigma_{xx} = 180$ MPa, $\sigma_{yy} = 90$ MPa, and $\sigma_{xy} = 50$ MPa.

Ans. $\sigma_1 = 202.3$ MPa, $\sigma_2 = 67.7$ MPa, $\sigma_3 = 0$, $\tau_{max} = 101.1$ MPa, $\tau_{oct} = 84.1$ MPa

2.24. The nonzero stress components are $\sigma_{xx} = -150$ MPa, $\sigma_{yy} = -70$ MPa, $\sigma_{zz} = 40$ MPa, and $\sigma_{xy} = -60$ MPa.

2.25. The nonzero stress components are $\sigma_{xx} = 80$ MPa, $\sigma_{yy} = -35$ MPa, $\sigma_{zz} = -50$ MPa, and $\sigma_{xy} = 45$ MPa.

Ans. $\sigma_1 = 95.5$ MPa, $\sigma_2 = -50.5$ MPa, $\sigma_3 = -50$ MPa, $\tau_{max} = 73.0$ MPa, $\tau_{oct} = 68.7$ MPa

2.26. The nonzero stress components are $\sigma_{xx} = 95$ MPa, $\sigma_{yy} = 0$, $\sigma_{zz} = 60$ MPa, and $\sigma_{xy} = -55$ MPa.

2.27. Let the state of stress at a point be given by $\sigma_{xx} = -120$ MPa, $\sigma_{yy} = 140$ MPa, $\sigma_{zz} = 66$ MPa, $\sigma_{xy} = 45$ MPa, $\sigma_{yz} = -65$ MPa, and $\sigma_{zx} = 25$ MPa. Determine the three principal stresses and directions associated with the three principal stresses.

Ans. $\sigma_1 = 180.2$ MPa, $\sigma_2 = 40.1$ MPa, $\sigma_3 = -134.3$ MPa
$l_1 = 0.0913$, $m_1 = 0.8740$, $n_1 = -0.4773$
$l_2 = 0.2584$, $m_2 = 0.4422$, $n_2 = 0.8589$
$l_3 = 0.9598$, $m_3 = -0.2062$, $n_3 = -0.1904$

2.28. Let the state of stress at a point be given by $\sigma_{xx} = 0$, $\sigma_{yy} = 100$ MPa, $\sigma_{zz} = 0$, $\sigma_{xy} = -60$ MPa, $\sigma_{yz} = 35$ MPa, and $\sigma_{zx} = 50$ MPa. Determine the three principal stresses.

2.29. Let the state of stress at a point be given by $\sigma_{xx} = 120$ MPa, $\sigma_{yy} = -55$ MPa, $\sigma_{zz} = -85$ MPa, $\sigma_{xy} = -55$ MPa, $\sigma_{yz} = 33$ MPa, and $\sigma_{zx} = -75$ MPa. Determine the three principal stresses and maximum shear stress.

Ans. $\sigma_1 = 162.5$ MPa, $\sigma_2 = -114.1$ MPa, $\sigma_3 = -68.4$ MPa, $\tau_{max} = 138.3$ MPa

2.30. Let the state of stress at a point be given by $\sigma_{xx} = -90$ MPa, $\sigma_{yy} = -60$ MPa, $\sigma_{zz} = 40$ MPa, $\sigma_{xy} = 70$ MPa, $\sigma_{yz} = -40$ MPa, and $\sigma_{zx} = -55$ MPa. Determine the three principal stresses and maximum shear stress.

2.31. Let the state of stress at a point be given by $\sigma_{xx} = -150$ MPa, $\sigma_{yy} = 0$, $\sigma_{zz} = 80$ MPa, $\sigma_{xy} = -40$ MPa, $\sigma_{yz} = 0$, and $\sigma_{zx} = 50$ MPa. Determine the three principal stresses and maximum shear stress.

Ans. $\sigma_1 = 91.2$ MPa, $\sigma_2 = 8.28$ MPa, $\sigma_3 = -169.5$ MPa, $\tau_{max} = 130.3$ MPa

2.32. **(a)** Solve Example 2.1 using Mohr's circle and show the orientation of the volume element on which the principal stresses act.

(b) Determine the maximum shear stress and show the orientation of the volume element on which it acts.

2.33. At a point on the flat surface of a member, load-stress relations give the following stress components relative to the (x, y, z) axes, where the z axis is perpendicular to the surface: $\sigma_{xx} = 240$ MPa, $\sigma_{yy} = 100$ MPa, $\sigma_{xy} = -80$ MPa, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$.

(a) Determine the principal stresses using Eq. (2.20), and then again using Eqs. (2.36) and (2.37).

(b) Determine the principal stresses using Mohr's circle and show the orientation of the volume element on which these principal stresses act.

(c) Determine the maximum shear stress and maximum octahedral shear stress.

Sections 2.5–2.8

2.34. The tension member in Fig. P2.34 has the following dimensions: $L = 5$ m, $b = 100$ mm, and $h = 200$ mm. The (x, y, z) coordinate axes are parallel to the edges of the member, with origin 0 located at the centroid of the left end. Under the deformation produced by load P, the origin 0 remains located at the centroid of the left end and the coordinate axes remain parallel to the edges of the deformed member. Under the action of load P, the bar elongates 20 mm. Assume that the volume of the bar remains constant with $\epsilon_{xx} = \epsilon_{yy}$.

(a) Determine the displacements for the member and the state of strain at point Q, assuming that the small-displacement theory holds.

(b) Determine ϵ_{zz} at point Q based on the assumption that displacements are not small.

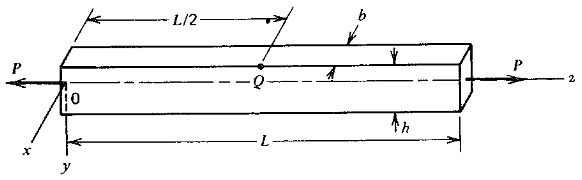

Figure P2.34

2.35. In many practical engineering problems, the state of strain is approximated by the condition that the normal and shear strains for some direction, say, the z direction, are zero, that is, $\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$ (plane strain). In Chapter 3, it is shown that analogously, $\epsilon_{zx} = \epsilon_{zy} = 0$, but $\epsilon_{zz} \neq 0$ for members made of isotropic materials and loaded such that the state of stress may be approximated by the condition $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$ (plane stress). Assume that ϵ_{xx}, ϵ_{yy}, and ϵ_{xy} for the (x, y) coordinate axes shown in Fig. P2.35 are known. Let the (X, Y) coordinate axes be defined by a counterclockwise rotation through angle θ as indicated in Fig. P2.35. Analogous to the

transformation for plane stress, show that the transformation equations of plane strain are $\epsilon_{XX} = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + 2\epsilon_{xy}\sin\theta\cos\theta$ and $\epsilon_{XY} = -\epsilon_{xx}\sin\theta\cos\theta + \epsilon_{yy}\sin\theta\cos\theta + \epsilon_{xy}(\cos^2\theta - \sin^2\theta)$. [See Eq. (2.30).]

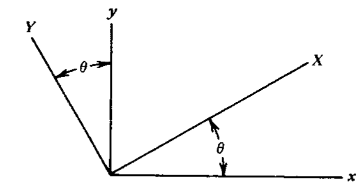

Figure P2.35

2.36. The square plate in Fig. P2.36 is loaded so that the plate is in a state of plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$).

(a) Determine the displacements for the plate given the deformations shown and the strain components for the (x, y) coordinate axes.

(b) Determine the strain components for the (X, Y) axes.

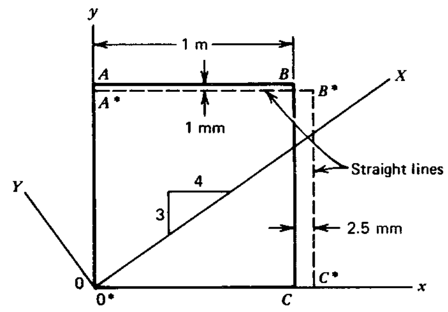

Figure P2.36

2.37. The square plate in Fig. P2.37 is loaded so that the plate is in a state of plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$.)

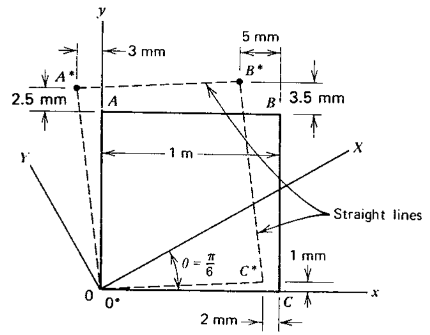

Figure P2.37

(a) Determine the displacements for the plate for the deformations shown and the strain components for the (x, y) coordinate axes.

(b) Determine the strain components for the (X, Y) axes.

Ans. (a) $u = -0.0020x - 0.0030y$, $v = 0.0010x + 0.0025y$, $\epsilon_{xx} = -0.0020$, $\epsilon_y = 0.0025$, $\gamma_{xy} = 2\epsilon_{xy} = -0.0020$;
(b) $\epsilon_{XX} = -0.00174$, $\epsilon_{YY} = 0.00224$, $\gamma_{XY} = 2\epsilon_{XY} = 0.00290$

2.38. Determine the orientation of the (X, Y) coordinate axes for principal directions in Problem 2.37. What are the principal strains?

2.39. The plate in Fig. P2.39 is loaded so that a state of plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$) exists.

(a) Determine the displacements for the plate for the deformations shown and the strain components at point B.

(b) Let the X axis extend from point 0 through point B. Determine ϵ_{XX} at point B.

Ans. (a) (dimensions in m) $u = 0.000667xy$, $v = 0.001333xy$, $\epsilon_{xx} = 0.00200$, $\epsilon_{yy} = 0.00200$, $\gamma_{xy} = 2\epsilon_{xy} = 0.00500$;
(b) $\epsilon_{XX} = 0.00400$

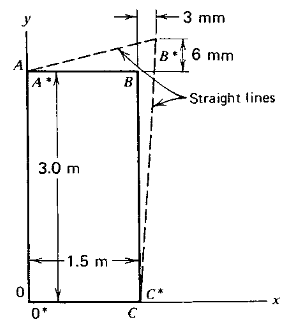

Figure P2.39

2.40. The nonzero strain components at a point in a loaded member are $\epsilon_{xx} = 0.00180$, $\epsilon_{yy} = -0.00108$, and $\gamma_{xy} = 2\epsilon_{xy} = -0.00220$. Using the results of Problem 2.35, determine the principal strain directions and principal strains.

2.41. Solve for the principal strains in Problem 2.40 by using Eqs. (2.77b) and (2.78).

Ans. $\epsilon_1 = 0.00217$, $\epsilon_2 = -0.00145$, $\epsilon_3 = 0$

2.42. Determine the principal strains at point E for the deformed parallelepiped in Example 2.4.

2.43. When solid circular torsion members are used to obtain material properties for finite strain applications, an expression for the engineering shear strain γ_{zx} is needed, where the (x, z) plane is a tangent plane and the z axis is parallel to

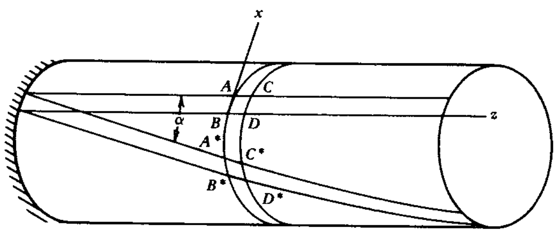

Figure P2.43

the axis of the member as indicated in Fig. P2.43. Consider an element $ABCD$ in Fig. P2.43 for the undeformed member. Assume that the member deforms such that the volume remains constant and the diameter remains unchanged. (This is an approximation of the real behavior of many metals.) Thus, for the deformed element $A^*B^*C^*D^*$, $A^*B^* = AB$, $C^*D^* = CD$, and the distance along the z axis of the member between the parallel curved lines A^*B^* and C^*D^* remains unchanged. Show that Eq. (2.71) gives the result $\gamma_{zx} = \tan\alpha$, where α is the angle between AC and A^*C^*, where $\gamma_{zx} = 2\epsilon_{zx}$ is defined to be the engineering shear strain.

2.44. A state of plane strain exists at a point in a member, with the nonzero strain components $\epsilon_{xx} = -2000\ \mu$, $\epsilon_{yy} = 400\ \mu$, and $\epsilon_{xy} = -900\ \mu$.

(a) Determine the principal strains in the (x, y) plane and the orientation of the rectangular element on which they act. (See Example 2.6.)

(b) Determine the maximum shear strain in the (x, y) plane and the orientation of the rectangular element on which it acts.

(c) Show schematically the deformed shape of a rectangular element in the reference orientation, along with the original undeformed element. (See Example 2.6.)

2.45. For the rectangular strain rosette, Fig. 2.20*b*, let arm a be directed along the positive x axis of axes (x, y).

(a) Show that the maximum principal strain is located at angle θ, counterclockwise to the x axis, where

$$\tan 2\theta = \frac{2\epsilon_b - \epsilon_a - \epsilon_c}{\epsilon_a - \epsilon_c}$$

(b) Show that the two principal surface strains ϵ_1 and ϵ_2 are given by

$$\epsilon_1 = \frac{\epsilon_a + \epsilon_c}{2} + R, \qquad \epsilon_2 = \frac{\epsilon_a + \epsilon_c}{2} - R$$

where

$$R = \tfrac{1}{2}[(\epsilon_a - \epsilon_c)^2 + (2\epsilon_b - \epsilon_a - \epsilon_c)^2]^{1/2}$$

(c) Construct the corresponding Mohr's circle for the rectangular rosette.

2.46. For the delta strain rosette, Fig. 2.20*a*, let arm a be directed along the positive x axis of axes (x, y).

(a) Show that the maximum principal strain is located at angle θ, counterclockwise to the x axis, where

$$\tan 2\theta = \frac{\sqrt{3}\,(\epsilon_b - \epsilon_c)}{2\epsilon_a - \epsilon_b - \epsilon_c}$$

(b) Show that the two principal surface strains ϵ_1 and ϵ_2 are given by

$$\epsilon_1 = \frac{\epsilon_a + \epsilon_b + \epsilon_c}{3} + R, \qquad \epsilon_2 = \frac{\epsilon_a + \epsilon_b + \epsilon_c}{3} - R$$

where

$$R = \tfrac{1}{3}[(2\epsilon_a - \epsilon_b - \epsilon_c)^2 + 3(\epsilon_b - \epsilon_c)^2]^{1/2}$$

(c) Construct the corresponding Mohr's circle for the delta rosette.

2.47. Let the arm a of a delta rosette, Fig. 2.20a, be directed along the positive x axis of axes (x, y). From measurements, $\epsilon_a = 2450\ \mu$, $\epsilon_b = 1360\ \mu$, and $\epsilon_c = -1310\ \mu$. Determine the two principal surface strains, the direction of the principal axes, and the associated maximum shear strain ϵ_{xy}.

2.48. Let the arm a of a rectangular rosette, Fig. 2.20b, be directed along the positive x axis of axes (x, y). Using Mohr's circle of strain, show that $2\epsilon_{XY} = \gamma_{XY} = 2\epsilon_b - \epsilon_a - \epsilon_c$.

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3

LINEAR STRESS-STRAIN-TEMPERATURE RELATIONS

In Chapter 2, we presented separate theories for stress and strain. These theories are based on the concept of a general continuum. Consequently, these theories are applicable to all continua. In particular, the theory of stress is based solely on the concept of force and the associated concept of force per unit area. Similarly, the theory of strain is based on geometrical concepts of infinitesimal line extensions and rotations between two infinitesimal lines. However, to relate the stress at a point in a material to the corresponding strain at that point, material properties are required. These properties enter into the stress-strain-temperature relations as material coefficients. The theoretical basis for these relations is the first law of thermodynamics.

In this chapter, we employ the first law of thermodynamics to derive linear stress-strain-temperature relations. In addition, certain concepts, such as complementary strain energy, that have application to nonlinear problems are introduced. These relations and concepts are utilized in many applications presented in subsequent chapters of this book.

3.1 FIRST LAW OF THERMODYNAMICS. INTERNAL-ENERGY DENSITY. COMPLEMENTARY INTERNAL-ENERGY DENSITY

The derivation of load-stress and load-deflection relations requires stress-strain relations that relate the components of the strain tensor to components of the stress tensor. The form of these relations depends on material behavior. In this book, we treat mainly materials that are isotropic, that is, at any point they have the same properties in all directions. Stress-strain relations for linearly elastic isotropic materials are well known and are presented in Sec. 3.4. Stress-strain relations may be treated theoretically by the use of the first law of thermodynamics, a precise statement of the law of conservation of energy. It is noted that the total amount of internal energy in a system is generally indeterminate. Hence, only changes of internal energy are measurable. These changes are determined by the