Chapter 27 Ordinary Differential Equations (ODE's)

Boundary-value and Eigen value problems

- What is the difference between Initial value problem (IVP) and Boundary-value problem (BVP)?

- Initial value problems, involves time. Initial values are always when x (or t) are equal to zero. Example:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$ both at $t=0$

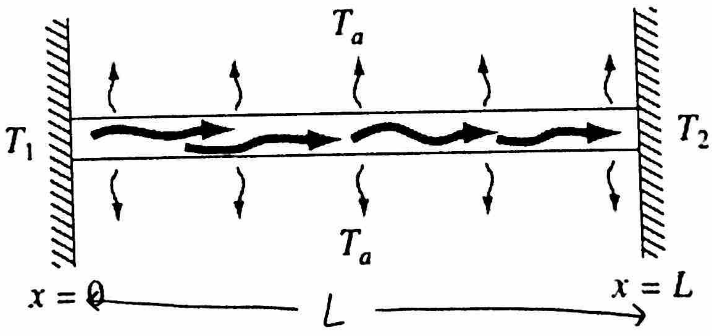
$$\ddot{x} = \frac{d^2x}{dt^2}, \quad \dot{x} = \frac{dx}{dt}$$

- Boundary value problems. conditions are specified at extreme points or "Boundaries". like beams, heat transfer. Example

ODE

$$\frac{d^2T}{dx^2} + \alpha(T_a - T) = 0$$

Heat transfer in a rod



Boundary conditions

$$T(0) = T_1$$
$$T(L) = T_2$$

α: heat coefficient

T_a: Surrounding temperature

27.1.1 The Shooting method

The shooting method is based on converting BVP to a system of differential equation with IVP.

Example

For the heat and temperature equation

$$\frac{d^2T}{dx^2} + \alpha(T_a - T) = 0 \quad , \quad \text{If} \quad l = 10 \text{ m}$$

$\alpha = 0.01 \ 1/m^2$ and $T_a = 20°C$. The boundary conditions are

$T(0) = 40°C$, $T(10) = 200°C$. Solve this equation

If step size is to be 2 m.

Solution

* We convert from BVP to IVP

$$\frac{dT}{dx} = z \quad \text{—} (1)$$

$$\frac{d^2T}{dx^2} = \frac{dz}{dx} = -\alpha(T_a - T) \quad \text{—} (2)$$

The initial condition for equation (1) :

$$T(0) = 40°C$$

But we need an initial condition for eq (2) $(z(0) = ?)$

Remember, From equation (1),

$$\frac{dT}{dx} = z \Rightarrow dT = z\,dx \xrightarrow{\text{Integrate}} T = zx$$

$$\text{or} \quad z(x) = \frac{T(x)}{x} \quad , \quad z(0) = \frac{T(0)}{0} = \frac{40}{0} = \infty$$

needs to be guessed

Now, lets guess, for example

$z(0) = 10$

$\Rightarrow \quad \frac{dT}{dx} = z \quad , \quad T(0) = 40^\circ C$

$\frac{dz}{dx} = -\alpha(T_a - T) \quad , \quad z(0) = 10 \ ^\circ C/m$

- Now, we solve this system of ODE's using RK4, (see 25.4), using $h=2$. Thus,:

$z_0 = z(0) = 10, \quad z_1 = z_0 + h, \ \dots, \ z_5 = z_4 + h = z_0 + 5h = 10$

we find $z_1, z_2, \dots, z_5$ and $T_1, T_2, \dots, T_5$

at $x = 10$, $T(10) = 168.38^\circ C$

- Remember from Boundary condition at $x = 10$, $T(10) = 200^\circ C$
- we need another guess for $z(0) = ?$ say, $z(0) = 20$, and resolve the system.

For this initial condition of z, $T(10) = 285.90 \ ^\circ C$

- Now,

$z_1(0) = 10 \longrightarrow T_1(10) = 168.38^\circ C$

$z(0) = z \ ? \longrightarrow T_z(10) = 200^\circ C$

$z_2(0) = 20 \longrightarrow T_2(10) = 285.90^\circ C$

Interpolation (Chapter 18)

$$\frac{z(0) - z_1(0)}{z_2(0) - z_1(0)} = \frac{T_z(10) - T_1(10)}{T_2(10) - T_1(10)}$$

$$z(0) = z = z_1(0) + (z_2(0) - z_1(0))\left(\frac{T_z(10) - T_1(10)}{T_2(10) - T_1(10)}\right)$$

substitute

$$Z = 10 + (20 - 10)\left(\frac{200 - 168.38}{285.90 - 168.38}\right)$$

$$Z = 12.70 \quad , \quad Z(0) = 12.70 .$$

Now, we use this initial condition to solve the system of ODE's.

x	$T(x)$
0	$T(0) = 40$
2	$T(2) = 65.95$
4	$T(4) = 93.75$
6	$T(6) = 124.50$
8	$T(8) = 159.45$
10	$T(10) = 200.$

* Shooting method can be used to solve non-linear ODE's. However, smaller step size is required

* In MATLAB, ode45 and ode23 functions can be used for solving systems of ODE's.

27.1.2 Finite-Difference method

For the equation,

$$\frac{d^2T}{dx^2} + \alpha(Ta - T) = 0 \qquad \text{— (1)}$$

Remember, from chapter 23, the 2nd derivative using centered-finite-divided difference method is

$$\frac{d^2T}{dx^2} = \frac{T(x_{i+1}) - 2T(x_i) + T(x_{i-1})}{h^2}$$

Back in eq(1)

$$\frac{T(x_{i+1}) - 2T(x_i) + T(x_{i-1})}{h^2} - \alpha(T(x_i) - T_a) = 0$$

$i = 1, 2, 3 \ldots$

or,

$$-T(x_{i-1}) + (2 + \alpha h^2)T(x_i) - T(x_{i+1}) = \alpha h^2 Ta$$

$$-T_{i-1} + (2 + \alpha h^2)T_i - T_{i+1} = \alpha h^2 Ta$$

we will get set of linear equations that canbe easily solved using Gauss, Gauss-Jordan or LU method

Example Use finite divided difference approach to solve

$$\frac{d^2T}{dx^2} - \alpha(T - T_a) = 0 \qquad T(0) = 40, \; T(10) = 200, \; T_a = 20, \; \alpha = 0.01, \; h = 2$$

$$-T_{i-1} + (2 + \alpha h^2) T_i - T_{i+1} = \alpha h^2 T_a$$

$i = 1$

$$-\underset{=40}{T_0} + (2 + \underset{0.01}{\alpha}\,\underset{(2)^2}{h^2}) T_1 - T_2 = \underset{0.01}{\alpha}\,\underset{(2)^2}{h^2}\,\underset{(20)}{T_a}$$

$$-40 + 2.04\,T_1 - T_2 = 0.8$$

$$\boxed{2.04\,T_1 - T_2 = 40.8} \quad — ①$$

$T_0 = T(0)$
$T_1 = T(2)$
$T_2 = T(4)$
$T_3 = T(6)$
$T_4 = T(8)$
$T_5 = T(10)$

$i = 2$

$$-T_1 + (2 + \alpha h^2) T_2 - T_3 = \alpha h^2 T_a$$

$$\boxed{-T_1 + 2.04\,T_2 - T_3 = 0.8} \quad -2$$

$i = 3$

$$-T_2 + (2 + \alpha h^2) T_3 - T_4 = \alpha h^2 T_a$$

$$\boxed{-T_2 + 2.08\,T_3 - T_4 = 0.8} \quad -3$$

$i = 4$

$$-T_3 + (2 + \alpha h^2) T_4 - T_5 = \alpha h^2 T_a, \qquad T_5 = T(10) = 200$$

$$\Rightarrow \boxed{-T_3 + 2.04\,T_4 = 200.8} \quad -4$$

Matrix Form

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{Bmatrix}$$

Solve $T_1 = 65.97$, $T_2 = 93.78$, $T_3 = 124.54$, $T_4 = 159.48$

* Practice problem : Use Finite-divided difference method to solve for the Column Buckling Problem as defined as

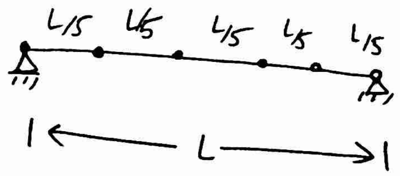
$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$, M = Coupling moment
E = Young's modulus
I = moment of Inertia

If $M = Py$, P = Force, y = location

$$\frac{d^2y}{dx^2} = \frac{Py}{EI} \Rightarrow \frac{d^2y}{dx^2} - \frac{Py}{EI} = 0$$

If the Column length is L, Use stepsize

$h = \frac{L}{5}$



* All previously considered Boundary conditions didn't include any derivatives - Dirichlet BC's.

* what if we have a derivative BC's ? (Neuman's BC's).

For example

$$\frac{d^2T}{dx^2} + \alpha (T_a - T) = 0 \quad , \quad \frac{dT}{dx}(0) = T_a' \quad T(L) = T_b$$

Using Centered F.D.D

$$- T_{i-1} + (2 + \alpha h^2) T_i - T_{i+1} = \alpha h^2 T_a$$

But for 1st derivative using Centered F.D.D

$$\frac{dT}{dx} = \frac{T_{i+1} - T_{i-1}}{2h} \Rightarrow T_{i-1} = T_{i+1} - 2h \frac{dT}{dX}$$

Thus,

$$\Rightarrow (2 + \alpha h^2) T_i - 2 T_{i+1} = \alpha h^2 T_a - 2h \frac{dT}{dX}$$

This includes the effect of derivative type BC. in the solution!