Chapter 25 Ordinary Differential Equations (ODE's)

Runge-Kutta Methods

For a differential equation:

$$\frac{dy}{dx} = f(x,y)$$

$$dy = f(x,y)\,dx \Rightarrow \text{Integrate} \Rightarrow y(x) = \int f(x,y)\,dx$$

↳ function

25.1 Euler's Method

$$y_{i+1} = y_i + \frac{dy}{dx} h \quad , \quad y_{new} = y_{old} + (\text{slope})(\text{step size})$$

$$\boxed{y_{i+1} = y_i + f(x_i, y_i)\,h}$$ Euler's Equation

$$y_{i+1} = y(x_{i+1}) \,,\quad y_i = y(x_i) \,,\quad x_{i+1} = x_i + h$$

Example Use Euler's method to integrate

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

from $x = 0$ to $x = 1$, use $h = 0.5$, $y(0) = 1$, $x_0 = 0$

Solution

$$y(x_{i+1}) = y(x_i) + f(x_i, y(x_i))\,h$$

$i = 0$

$$y(x_1) = y(x_0) + f(x_0, y(x_0))\,h$$

$x_0 = 0$
$x_1 = x_0 + h = 0.5$

$$y(0.5) = y(0) + f(0, y(0))\,h \quad , \quad f(0, y(0)) = f(0,1)$$

$$= 1 + (8.5)(0.5) \Rightarrow y(0.5) = 5.25$$

$$f(x,y) = \frac{dy}{dx} \Rightarrow \left.\frac{dy}{dx}\right|_{0,1} = 8.5$$

$\underline{i=1}$ $\quad y(x_2) = y(x_1) + f(x_1, y(x_1))h$ $\qquad x_1 = 0.5,\ x_2 = x_1 + h = 1$

$$y(1) = y(0.5) + f(0.5, y(0.5))h \quad , \ f(0.5, 5.25)$$

$$= 5.25 + f(0.5, 5.25)h$$

$$= 5.25 + (1.25)(0.5) \Rightarrow \boxed{y(1) = 5.875}$$

$$\left.\frac{dy}{dx}\right|_{0.5, 5.25} = 1.25$$

For more details, see example 25.1 (Page 710) from your textbook

- Remember Taylor Series:

$$y_{i+1} = y_i + y_i' h + \frac{y'' h^2}{2!} + \frac{y''' h^3}{3!} + \cdots + \frac{y^{(n)} h^n}{n!} + \underbrace{R_n}_{\text{Remainder}}$$

But, $\quad y_i' = \frac{dy}{dx} = f(x_i, y_i)$

Thus,

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)h^2}{2} + \cdots + \frac{f^{(n-1)}(x_i, y_i)h^n}{n!} + O(h^{n+1})$$

$O(h^{n+1})$ = Local truncation error

→ step size

→ n

- Euler's Equation $\Rightarrow y_{i+1} = y_i + f(x_i, y_i)h$

- True error E_t = true − Approx $\Rightarrow$ $\underbrace{E_t = \frac{f'(x_i, y_i)h^2}{2} + \cdots + O(h^{n+1})}_{\text{True local truncation error}}$

$$E_a = \frac{f'(x_i, y_i)}{2} h^2$$ ← Approximate local truncation Error

25.2 How to improve Euler's method ?

- Huen's method
- Mid-point method

Both methods above, require two steps to estimate the ODE solution (1- Predictor, 2- corrector)

* Huen's method

$$\frac{dy}{dx} = f(x, y)$$

Predictor — ① $y^0_{i+1} = y_i + f(x_i, y_i)h$

corrector ② $y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})}{2} h$

* Mid-point method

$$\frac{dy}{dx} = f(x, y)$$

Predictor ① $y_{i+1/2} = y_i + f(x_i, y_i)\frac{h}{2}$

② $y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})h$

$$x_{i+1/2} = x_i + \frac{1}{2}h$$

Example $\frac{dy}{dx} = 4e^{0.8x} - 0.5y$, Integrate from $x=0$ to $x=1$ using $h=1$ and $y(0)=2$ in

(a) Huen's method

(b) Mid-point method

Solution (a) Huen's method

$$y^0_{i+1} = y_i + f(x_i, y_i)h$$

$$y_{i+1} = y_i + \frac{f(x_i,y_i) + f(x_{i+1}, y^0_{i+1})}{2} h$$

$i=0$

$$y^0_1 = y_0 + f(x_0, y_0)h$$

$$y^0(x_1) = y(x_0) + f(x_0, y(x_0))h \quad , x_0 = 0 \quad , x_1 = x_0 + h = 1$$

$$y^0(1) = y(0) + f(0, y(0))h \quad , f(0, y(0)) = f(0,2)$$

$$y^0(1) = 2 + 3(1) \Rightarrow y^0(1) = 5 \quad , f(0,2) = 4e^{0.8(0)} - (0.5)(2) = 3$$

$$y_1 = y_0 + \frac{f(x_0, y_0) + f(x_1, y^0_1)}{2} h$$

$$y(x_1) = y(x_0) + \frac{f(x_0, y(x_0)) + f(x_1 + \overbrace{y^0(x_1)}^{= y^0(1)})}{2} h$$

$$y(1) = y(0) + \frac{f(0,2) + f(1,5)}{2} h \quad ,$$

$$= 2 + \frac{3 + 6.4}{2}(1) \Rightarrow y(1) = 6.701$$

$$f(1,5) = 4e^{0.8(1)} - (0.5)(1)$$

$$= 6.4$$

(b) Mid-point method

$$y_{i+1/2} = y_i + f(x_i, y_i)\frac{h}{2}$$

$$y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})h$$

$i=0$

$$y_{1/2} = y_0 + f(x_0, y_0)\frac{h}{2}$$

$$y(x_{1/2}) = y(x_0) + f(x_0, y(x_0))\frac{h}{2} \qquad x_0 = 0,\ x_{1/2} = x_0 + \frac{h}{2} = 0.5$$

$$y(0.5) = y(0) + f(0, y(0))\frac{h}{2} \qquad ,\ y(0) = 2$$

$$= y(0) + f(0,2)\frac{h}{2}$$

$$= 2 + (3)\left(\frac{1}{2}\right) \Rightarrow y(0.5) = 3.5$$

$$y_1 = y_0 + f(x_{1/2}, y_{1/2})h$$

$$y(x_1) = y(x_0) + f(x_{1/2}, y(x_{1/2}))h \qquad x_0 = 0,\ x_1 = x_0 + h = 1$$

$$y(1) = y(0) + f(0.5, y(0.5))h \qquad ,\ y(0.5) = 3.5$$

$$= y(0) + f(0.5, 3.5)h$$

$$= 2 + (4.22)(1) \Rightarrow y(1) = 6.22$$

$$f(0.5, 3.5) = 4e^{(0.8)(1/2)} - (0.5)(3.5)$$

$$= 4.22$$

Chapter 25: ODE's

25.3 Runge-Kutta methods (RK methods)

- First order RK (RK1) → Euler's method
- Second order RK (RK2)
- Third order RK (RK3)
- Fourth order RK (RK4)

* First order RK (RK1)

$$\frac{dy}{dx} = f(x, y)$$

$$y_{i+1} = y_i + k_1 h \quad , \quad k_1 = f(x_i, y_i) \quad , \quad h = \text{Step Size}$$

Improvements → Heun's / Mid-point methods

* Second order RK (RK2)

$$\frac{dy}{dx} = f(x, y)$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) h$$

$$k_1 = f(x_i, y_i) \quad , \quad k_2 = f(x_i + p_1 h \, , \, y_i + q_{11} k_1 h)$$

a_1, a_2, p_1 and q_{11} ⇒ Constants

See Box 25.1 page 731 For more details

$$\left.\begin{aligned} a_1 + a_2 &= 1 \\ a_2 p_1 &= \frac{1}{2} \\ a_2 q_{11} &= \frac{1}{2} \end{aligned}\right\} \Rightarrow \text{3 equations and 4 unknowns}$$

⇒ Assume a_2 and find a_1, p_1, and q_{11}

* Huen's method ($a_2 = 1/2$)

$\Rightarrow a_1 = 1/2 \;,\; P_1 = 1 \;,\; q_{11} = 1$

$$\Rightarrow \begin{cases} y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2)h \\ K_1 = f(x_i, y_i) \;,\; K_2 = f(x_i + h, y_i + K_1 h) \end{cases}$$

* Mid-point method ($a_2 = 1$)

$\Rightarrow a_1 = 0 \;,\; P_1 = 1/2 \;,\; q_{11} = 1/2$

$$\Rightarrow \begin{cases} y_{i+1} = y_i + K_2 h \\ K_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h) \;,\; K_1 = f(x_i, y_i) \end{cases}$$

* Raltson's method ($a_2 = 2/3$)

$a_1 = \frac{1}{3} \;,\; P_1 = 3/4$ and $q_{11} = 3/4$

$$\Rightarrow \begin{cases} y_{i+1} = y_i + \frac{1}{3}(K_1 + 2K_2)h \\ K_1 = f(x_i, y_i) \;,\; K_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}K_1 h) \end{cases}$$

* Third order RK (RK3)

$$\frac{dy}{dx} = f(x, y)$$

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 4K_2 + K_3)h$$

$$K_1 = f(x_i, y_i), \quad K_2 = f\left(x_i + \tfrac{1}{2}h,\ y_i + \tfrac{1}{2}K_1 h\right)$$

$$K_3 = f(x_i + h,\ y_i - K_1 h + 2K_2 h)$$

* Fourth order RK (RK4)

$$\frac{dy}{dx} = f(x, y)$$

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)h$$

$$K_1 = f(x_i, y_i), \quad K_2 = f\left(x_i + \tfrac{1}{2}h,\ y_i + \tfrac{1}{2}K_1 h\right)$$

$$K_3 = f\left(x_i + \tfrac{1}{2}h,\ y_i + \tfrac{1}{2}K_2 h\right), \quad K_4 = f(x_i + h,\ y_i + K_3 h)$$

Example: $f(x,y) = \frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$

Solve the above ODE using RK4 from $x=0$ to $x=0.5$

Use $h = 0.5$ and $y(0) = 1$

Solution

$i=0$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\,h$$

$$y_0 = y(x_0) = y(0) = 1 \,, \quad k_1 = f(x_0, y_0) = f(x_0, y(x_0)) = f(0,1) = 8.5$$

$$k_2 = f\Big(x_0 + \frac{h}{2},\ \underbrace{y_0}_{=y(x_0)} + \frac{1}{2}k_1 h\Big) = f\Big(0 + \frac{0.5}{2},\ 1 + \frac{1}{2}(8.5)(0.5)\Big)$$

$$= f(0.25, 3.125) = 4.22 \,, \quad k_2 = 4.22$$

$$k_3 = f\Big(x_0 + \frac{1}{2}h,\ \underbrace{y_0}_{=y(x_0)} + \frac{1}{2}k_2 h\Big) = f\Big(0 + \frac{1}{2}(0.5),\ 1 + \frac{1}{2}(4.22)(0.5)\Big)$$

$$= f(0.25, 2.06) \Rightarrow k_3 = 4.22$$

$$k_4 = f(x_0 + h,\ \underbrace{y_0}_{=y(x_0)} + k_3 h) = f(0.5,\ 1 + (4.22)(0.5))$$

$$= f(0.5, 3.11) \Rightarrow k_4 = 3.11$$

$$\Rightarrow y(x_1) = \underbrace{y(x_0)}_{=1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\,h$$

$$y(0.5) = y(0) + \frac{1}{6}(8.5 + (2)(4.22) + (2)(4.22) + 3.11)(0.5)$$

$$y(0.5) = 3.22$$

Chapter 25 ODE's

25.4 Systems of Equations

A system of ODE's can be represented as:

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n)$$

- To solve such kind of problems, we need (n) initial conditions
- We can apply all methods we have learnt previously to solve such system. However, for each y, we need a complete solution
- Euler's method

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2) \quad , \quad \frac{dy_2}{dx} = f_2(x, y_1, y_2)$$

For each y, we need a solution, like

$$y_1(x_{i+1}) = y_1(x_i) + f_1(x_i, y_1(x_i), y_2(x_i))\,h$$

$$y_2(x_{i+1}) = y_2(x_i) + f_2(x_i, y_1(x_i), y_2(x_i))\,h$$

* This can also be applied in all RK methods, like let's take RK3, for example

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n) \quad \cdots \quad \frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n)$$

RK3

$$y_1(x_{i+1}) = y_1(x_i) + \frac{1}{6}(K_{1,1} + 4K_{2,1} + K_{3,1})h$$

K_{ij} i (1,2,3) j (1,2,...,n)

$$y_2(x_{i+1}) = y_2(x_i) + \frac{1}{6}(K_{1,2} + 4K_{2,2} + K_{3,2})h$$

$$\vdots$$

$$y_n(x_{i+1}) = y_n(x_i) + \frac{1}{6}(K_{1,n} + 4K_{2,n} + K_{3,n})h$$

$$K_{1,1} = f_1(x, y_1(x_i), y_2(x_i), \dots, y_n(x_i))$$

$$K_{1,2} = f_2(x, y_1(x_i), y_2(x_i), \dots, y_n(x_i))$$

$$\vdots$$

$$K_{1,n} = f_n(x, y_1(x_i), y_2(x_i), \dots, y_n(x_i))$$

$$K_{2,1} = f_1(x_i + \tfrac{1}{2}h, y_1(x_i) + \tfrac{1}{2}K_{1,1}h, y_2(x_i) + \tfrac{1}{2}K_{1,2}h, \dots, y_n(x_i) + \tfrac{1}{2}K_{1,n}h)$$

$$K_{2,2} = f_2(x_i + \tfrac{1}{2}h, y_1(x_i) + \tfrac{1}{2}K_{1,1}h, y_2(x_i) + \tfrac{1}{2}K_{1,2}h, \dots, y_n(x_i) + \tfrac{1}{2}K_{1,n}h)$$

$$\vdots$$

$$K_{2,n} = f_n(x_i + \tfrac{1}{2}h, y_1(x_i) + \tfrac{1}{2}K_{1,1}h, y_2(x_i) + \tfrac{1}{2}K_{1,2}h, \dots, y_n(x_i) + \tfrac{1}{2}K_{1,n}h)$$

$$K_{3,1} = f_1(x_i + h, y_1(x_i) - K_{1,1}h + 2K_{2,1}h, \dots, y_n(x_i) - K_{1,n}h + 2K_{2,n}h)$$

$$K_{3,2} = f_2(x_i + h, y_1(x_i) - K_{1,1}h + 2K_{2,1}h, \dots, y_n(x_i) - K_{1,n}h + 2K_{2,n}h)$$

$$\vdots$$

$$K_{3,n} = f_n(x_i + h, y_1(x_i) - K_{1,1}h + 2K_{2,1}h, \dots, y_n(x_i) - K_{1,n}h + 2K_{2,n}h)$$

Practice problem : For the system of ODE's we discussed, Please write the general soltn for y_i $(i=1,2,\dots,n)$ using the methods we discussed

1- Euler's rule

1.1- Euler's with Huen's method

1.2- Euler's with Mid point

2 - RK2

→ RK2 with Huen's method

→ RK2 with mid-point method

→ RK2 with Raltson's method

3- RK 3 (Already done)

4- RK4

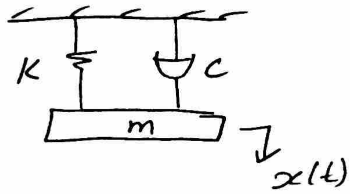
* In real-life applications, we commonly have higher order ODE's

* Till now, we only Discussed First order ODE's

* To solve higher order ODE's, we need to transform them to system of 1st order ODE's

<u>Example</u>, For a spring-mass-damper system vibration, the governing equation is

$$m\ddot{x} + c\dot{x} + kx = 0 \quad , \quad \ddot{x} = \frac{d^2x}{dt^2}$$
$$\dot{x} = \frac{dx}{dt}$$



To transform this second order ODE to system of 1st order ODE's :

$$y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{dy}{dt} + cy + kx = 0 \Rightarrow \frac{dy}{dt} = -\frac{cy+kx}{m}$$

$\Rightarrow$ <u>system of ODE's became</u>

$$\frac{dx}{dt} = f_1(t, y, x)$$

$$\frac{dy_n}{dx} = f(x, y_1, y_2)$$

t y x

$$\frac{dy}{dt} = f_2(t, y, x)$$

now, we can solve for y and x. (y_1, y_2)

we need to Initial Conditions to solve (will be given)