Chapter 11: Special matrices & Gauss-Siedel

* Gauss-Siedel is a method for solving systems of linear Equations
* Needs iterations

For a system of linear equation

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = C_1 \quad \text{— Eq(1)}$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = C_2 \quad \text{— Eq(2)}$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = C_3 \quad \text{— Eq(3)}$$

Gauss-Siedel :-

$$x_1 = \frac{C_1 - a_{12} x_2 - a_{13} x_3}{a_{11}} \quad (a) \quad \text{, From Eq(1)}$$

$$x_2 = \frac{C_2 - a_{21} x_1 - a_{23} x_3}{a_{22}} \quad (b) \quad \text{, From Eq(2)}$$

$$x_3 = \frac{C_3 - a_{31} x_1 - a_{32} x_2}{a_{33}} \quad (c) \quad \text{, From Eq(3)}$$

* Conditions

1. $a_{ii} \neq 0$, $a_{11}, a_{22}, a_{33} \neq 0$

2. $|a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}} |a_{ij}| \Rightarrow$
$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

Solution procedure:

① Use initial guesses $x_1 = x_2 = x_3 = 0$

② Substitute in Eq(a) $\Rightarrow$ $x_1 = \dfrac{c_1}{a_{11}}$

③ Substitute in Eq(b) $\Rightarrow$ $x_2 = \dfrac{c_2 - a_{21}x_1 - a_{23}\overset{=0}{x_3}}{a_{22}}$

④ Substitute in Eq(c) $\Rightarrow$ $x_3 = \dfrac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$, x_1 from (2)

⑤ Repeat steps 2,3 and 4 till we reach $\epsilon_a \leq \epsilon_s$

$$\epsilon_{a,i} = \left| \frac{x_i^{j} - x_i^{j-1}}{x_i^{j}} \right| 100\% \quad , i = 1,2,3$$

j: Iteration #

$$= \left| \frac{\text{Present} - \text{Previous}}{\text{Present}} \right| 100\%$$

Example:- Use Gauss-Siedel to Solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Solution

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

Now, we do iterations

Iteration # 1

① $x_1 = x_2 = x_3 = 0$

② $x_1 = \dfrac{7.85}{3} \Rightarrow x_1 = 2.6167$

③ $x_2 = \dfrac{-19.3 - 0.1x_1}{7} = \dfrac{-19.3 - (0.1)(2.6167)}{7} \Rightarrow x_2 = -2.7945$

④ $x_3 = \dfrac{71.4 - 0.3x_1 + 0.2x_2}{10} = \dfrac{71.4 - (0.3)(2.6167) + (0.2)(-2.7945)}{10}$

$$\Rightarrow x_3 = 7.0056$$

* Iteration # 2

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} = \frac{7.85 + (0.1)(-2.7945) + (0.2)(7.0056)}{3}$$

$$x_1 = 2.9906$$, we use the updated values

$$\epsilon_{a,1} = \left|\frac{2.9906 - 2.6167}{2.9906}\right| 100\% = 12.5\%$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} = \frac{-19.6 - (0.1)(2.9906) + (0.3)(7.0056)}{7}$$

$$x_2 = -2.4996$$

$$\epsilon_{a,2} = \left|\frac{-2.4996 - (-2.7945)}{-2.4996}\right| 100\% = 11.8\%$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} = \frac{71.4 - (0.3)(2.9906) + (0.2)(-2.4996)}{10}$$

$$x_3 = 7.0003$$

$$\epsilon_{a,3} = \left|\frac{7.0003 - 7.0056}{7.0003}\right| 100\% = 0.08\%$$

Repeat this till $\epsilon_a \leq \epsilon_s$

* Jacobi Iterations

* Very similar to Gauss-Siedel.

* For the same system we used previously,

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \qquad (a)$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \qquad (b)$$

$$x_3 = \frac{c_3 - a_{31}x_1 - a_{32}x_2}{a_{33}} \qquad (c)$$

Condition $\quad a_{ii} \neq 0$ and $|a_{ii}| > \sum |a_{ij}|$

Solution procedure

① Initial guesses $x_1 = x_2 = x_3 = 0$

② substitute in Eq (a), (b) and (c)

$$x_1 = \frac{c_1}{a_{11}}, \quad x_2 = \frac{c_2}{a_{22}}, \quad x_3 = \frac{c_3}{a_{33}}$$

③ Repeat till we reach desired ε_a

Example: Use Jacobi iterations method to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$

$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Solution

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3}$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7}$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

Now, we do iterations

① $x_1 = x_2 = x_3 = 0$

② $$\left.\begin{aligned} x_1 &= \frac{7.85}{3} = 2.6167 \\ x_2 &= \frac{-19.3}{7} = -2.7571 \\ x_3 &= \frac{71.4}{10} = 7.14 \end{aligned}\right\} \text{Iteration \# 1}$$

Iteration # 2

$$x_1 = \frac{(7.85) + (0.1)(-2.7571) + (0.2)(7.14)}{3} \Rightarrow x_1 = 3.0008$$

we use values from previous iteration

$$\varepsilon_{a,1} = \left| \frac{3.0008 - 2.6167}{3.0008} \right| 100\% = 12.8\%$$

$$x_2 = \frac{-19.3 - (0.1)(2.6167) + (0.3)(7.14)}{7} \Rightarrow x_2 = -2.4885$$

$$\varepsilon_{a,2} = \left| \frac{-2.4885 - (-2.7571)}{-2.4885} \right| 100\% = 10.8\%$$

$$x_3 = \frac{71.4 + (0.2)(\overset{x_2}{-2.7571}) - (0.3)(\overset{x_1}{2.6167})}{10} \Rightarrow x_3 = 7.0064$$

$$\varepsilon_{a,3} = \left| \frac{7.0064 - 7.14}{7.0064} \right| 100\% = 1.9\%$$

Repeat to reach desired ε_a.

★ Summary

① Gauss-Siedel

$$x_1^{j} = \frac{c_1 - a_{12} x_2^{j-1} - a_{13} x_3^{j-1}}{a_{11}}$$

$$x_2^{j} = \frac{c_2 - a_{21} x_1^{j} - a_{23} x_3^{j-1}}{a_{22}}$$

$$x_3^{j} = \frac{c_3 - a_{31} x_1^{j} - a_{32} x_2^{j}}{a_{33}}$$

② Jacobi

$$x_1^{j} = \frac{c_1 - a_{12} x_2^{j-1} - a_{13} x_3^{j-1}}{a_{11}}$$

$$x_2^{j} = \frac{c_2 - a_{21} x_1^{j-1} - a_{23} x_3^{j-1}}{a_{22}}$$

$$x_3^{j} = \frac{c_3 - a_{31} x_1^{j-1} - a_{32} x_2^{j-1}}{a_{33}}$$

j is the iteration #.

End of chapter 11