Chapter 6 : 6.6 System of Non-linear Equations

For System of Non-linear Equations, like :

$$\left.\begin{aligned} u(x,y) &= x^2 + xy - 10 \\ v(x,y) &= y + 3xy^2 - 57 \end{aligned}\right\} \text{ Solve for } x \text{ and } y$$

- We can use Newton-Raphson method for non-linear equations, as:

$$x_{i+1} = x_i - \frac{u_i \frac{\partial v_i}{\partial y} - v_i \frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x} \cdot \frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y} \cdot \frac{\partial v_i}{\partial x}}$$

$$i = 0, 1, 2, \ldots$$

$$y_{i+1} = y_i - \frac{v_i \frac{\partial u_i}{\partial x} - u_i \frac{\partial v_i}{\partial x}}{\frac{\partial u_i}{\partial x} \cdot \frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y} \cdot \frac{\partial v_i}{\partial x}}$$

- We need
  1. Initial guesses for x and y
  2. Iterations till we reach $\epsilon_{a,x}$ and $\epsilon_{a,y}$

Example For the system below, Solve for x and y

Given $x_0 = 1.5$, $y_0 = 3.5$

$$u(x,y) = x^2 + xy - 10$$
$$v(xy) = y + 3xy^2 - 57$$

Solution

$i = 0$

$$x_1 = x_0 - \frac{u_0 \frac{\partial v_0}{\partial y} - v_0 \frac{\partial u_0}{\partial y}}{\frac{\partial u_0}{\partial x} \cdot \frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial y} \cdot \frac{\partial v_0}{\partial x}}$$

$$y_1 = y_0 - \frac{v_0 \frac{\partial u_0}{\partial x} - u_0 \frac{\partial v_0}{\partial x}}{\frac{\partial u_0}{\partial x} \cdot \frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial y} \cdot \frac{\partial v_0}{\partial x}}$$

$$\frac{\partial u}{\partial x} = 2x + y \quad , \quad \frac{\partial u}{\partial y} = x$$

$$\frac{\partial v}{\partial x} = 3y^2 \quad , \quad \frac{\partial v}{\partial y} = 6xy + 1$$

$$u_0 = u(x_0, y_0) = u(1.5, 3.5) = (1.5)^2 + (1.5)(3.5) - 10 = -2.5$$
$$v_0 = v(x_0, y_0) = v(1.5, 3.5) = (3.5) + 3(1.5)(3.5)^2 - 57 = 1.625$$

$$\frac{\partial u_0}{\partial x} = 2x_0 + y_0 = (2)(1.5) + 3.5 = 6.5$$

$$\frac{\partial u_0}{\partial y} = x_0 = 1.5$$

$$\frac{\partial v_0}{\partial x} = 3y_0^2 = (3)(3.5)^2 = 36.75$$

$$\frac{\partial v_0}{\partial y} = 6x_0 y_0 + 1 = (6)(1.5)(3.5) + 1 = 32.5$$

$$\Rightarrow x_1 = 1.5 - \frac{(-2.5)(32.5) - (1.625)(1.5)}{(6.5)(32.5) - (1.5)(36.75)} = 2.0360$$

$$\Rightarrow y_1 = 3.5 - \frac{(1.625)(6.5) - (-2.5)(36.75)}{(6.5)(32.5) - (1.5)(36.75)} = 2.8439$$

And we continue doing iterations to reach

$\epsilon_{a,x}$ and $\epsilon_{a,y}$.

- For systems with any number of non-linear equations, this method can be generalized (Chapter 9 - Section 9.6)

$$f_1(x_1, x_2, \dots, x_n) = 0$$
$$f_2(x_1, x_2, \dots, x_n) = 0$$
$$\vdots$$
$$f_n(x_1, x_2, \dots, x_n) = 0$$

The solution $\{X_{i+1}\} = \{x_{1,i+1}, x_{2,i+1}, \dots, x_{n,i+1}\}$, can be obtained as:

$$[Z]\{X_{i+1}\} = -\{F_i\} + [Z]\{X_i\}, \quad i = 0, 1, 2, \dots$$

where $\{F_i\} = \begin{Bmatrix} f_{1,i} \\ f_{2,i} \\ \vdots \\ f_{n,i} \end{Bmatrix}$, $\{X_i\} = \begin{Bmatrix} x_{1,i} \\ x_{2,i} \\ \vdots \\ x_{n,i} \end{Bmatrix}$

This system can be finally solved using
- Gauss - Elimi.
- Gauss - Jordan
- LU method

$$[Z] = \begin{bmatrix} \frac{\partial f_{1,i}}{\partial x_1} & \frac{\partial f_{1,i}}{\partial x_2} & \cdots & \frac{\partial f_{1,i}}{\partial x_n} \\ \frac{\partial f_{2,i}}{\partial x_1} & \frac{\partial f_{2,i}}{\partial x_2} & \cdots & \frac{\partial f_{2,i}}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_{n,i}}{\partial x_1} & \frac{\partial f_{n,i}}{\partial x_2} & \cdots & \frac{\partial f_{n,i}}{\partial x_n} \end{bmatrix}$$