* Roots of Equations

$$f(x) = ax^2 + bx + C$$

Roots $\quad x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$f(x) = \sin(x) + \frac{1}{x} + e^{x^2}$, Find roots? We need numerical methods

Roots of Equations

Chapter 5
Bracketing methods

→ we need interval to find roots

Chapter 6
Open methods

→ we only one value (initial guess) to Find roots

1- Bisection Method

2- False-Position Method

1- Simple fixed-point

2- Newton-Raphson

3- Secant and modified Secant

4- Multiple roots (NR)

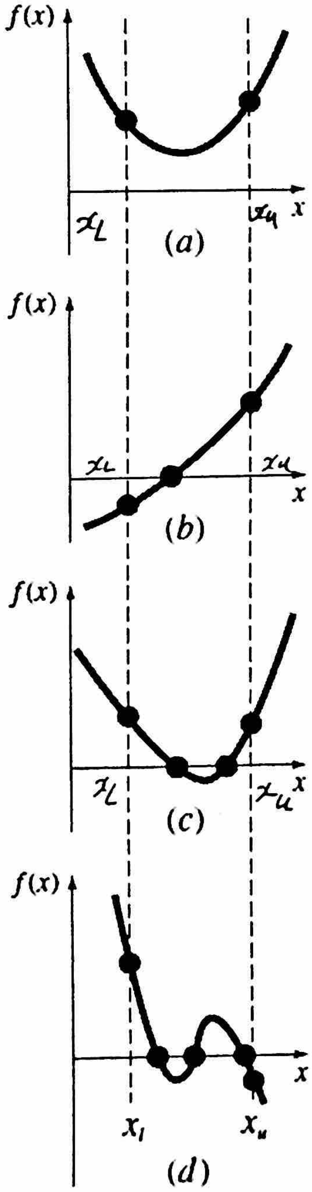
Chapter 5 : Bracketing methods

* If a function $f(x)$ has a root (x_r) lies in the interval $[x_L, x_u]$ and $f(x_u) \cdot f(x_L) < \overset{-ve}{0}$, then the Bisection method computes x_r, as:

$$x_r = \frac{x_u + x_L}{2}$$

x_L = Lower

x_u = Upper



$f(x_L) \cdot f(x_u) = (+)(+) = (+) \rightarrow$ No roots

$f(x_L) \cdot f(x_u) = (-)(+) = (-) \rightarrow$ one root

$f(x_L) \cdot f(x_u) = (+)(+)$ or $= (-)(-)$ $= (+)$ No roots or Even # of roots

$f(x_L) \cdot f(x_u) = (+)(-) = (-)(+) = (-) \rightarrow$ one root or odd # of roots

Solution procedure

① We have x_L and x_u, make sure

$f(x_L) \cdot f(x_u) < 0$, If not, we cannot use Bisection method (No solution)

② Find $x_r = \dfrac{x_L + x_u}{2}$

③ Compute $f(x_r)$, and:

→ If $f(x_L) \cdot f(x_r) = 0$, then x_r = exact root

→ If $f(x_r) \cdot f(x_L) > 0$ (+ve), $x_L = x_r$ and back to step 2

→ If $f(x_r) \cdot f(x_L) < 0$ (-ve), $x_u = x_r$ and back to step 2

This method needs <u>Iterations</u>

Example: $f(x) = e^{+x} - 2$, $x_L = 0$ and $x_u = 1$

- Find x_r for $\epsilon_a < |5\%|$

Solution

Iteration ①

- check $f(x_L) \cdot f(x_u) < 0$ $\Rightarrow$ $f(x_L) = f(0) = 1 - 2 = -1$

$f(x_u) = f(1) = e^{1} - 2 = 0.73$

$\Rightarrow f(x_L) \cdot f(x_u) = (-)(+) = (-)$

- $x_r = \dfrac{x_L + x_u}{2} = \dfrac{0+1}{2} \Rightarrow x_r = 0.5$

- $f(x_r) = f(0.5) = e^{+0.5} - 2 = -0.35$

$f(x_L) \cdot f(x_r) = (-)(-) = + \Rightarrow x_L = x_r = 0.5$

Iteration 2

$x_L = 0.5$, $x_u = 1$ (no need to check $f(x_L) \cdot f(x_u) < 0$)

$x_r = \dfrac{x_L + x_u}{2} = \dfrac{0.5 + 1}{2} = 0.75$

$f(x_r) = f(0.75) = e^{+0.75} - 2 = 0.12$ (+)

$f(x_L) \cdot f(x_r) = f(0.5) \cdot f(0.75) = (-)(+) = (-)$

$x_u = x_r = 0.75$

Iteration 3

$x_L = 0.5 \qquad x_u = 0.75$

$$x_r = \frac{x_L + x_u}{2} = \frac{0.5 + 0.75}{2} = 0.625$$

$$f(x_r) = f(0.625) = e^{+0.625} - 2 = -0.13$$

$$f(x_L) \cdot f(x_r) = f(0.5) \cdot f(0.625) = (-)(-) = + \qquad x_L = x_r = 0.625$$

Till when do we need to find iterations?

Till we reach ε_a. In this example, $\varepsilon_a < 5\%$

Iteration	x_L	x_u	x_r	Sign $f(x_L)f(x_r)$	ε_a
1	0	1	0.5	+	N/A
2	0.5	1	0.75	−	$\varepsilon_a = \left\lvert \frac{0.75 - 0.5}{0.75} \right\rvert 100\% = 33.3\%$
3	0.5	0.75	0.625	+	$\varepsilon_a = \left\lvert \frac{0.625 - 0.75}{0.625} \right\rvert 100\% = 20\%$
4	0.625	0.75	0.6875	+	$\varepsilon_a = \left\lvert \frac{0.6875 - 0.625}{0.6875} \right\rvert 100\% = 9\%$
5	0.6875	0.75	0.71875	(−)	$\varepsilon_a = \left\lvert \frac{0.71875 - 0.6875}{0.71875} \right\rvert 100\% = 4.3\%$

$$\varepsilon_a = \left\lvert \frac{\text{Present value of } x_r - \text{Previous value of } x_r}{\text{Present value of } x_r} \right\rvert 100\%$$

$$\boxed{x_r = 0.71875}$$

Iteration 4

$x_l = 0.625$, $x_u = 0.75$

$$x_r = \frac{x_L + x_u}{2} = \frac{0.625 + 0.75}{2} = 0.6875$$

$$f(x_r) = f(0.6875) = e^{0.6875} - 2 = -0.01$$

$$f(x_L) \cdot f(x_r) = (-)(-) = + \qquad x_L = x_r = 0.6875$$

Iteration 5

$x_L = 0.6875$, $x_u = 0.75$

$$x_r = \frac{x_L + x_u}{2} = 0.71875$$

- Note:

approx. error

In Bisection method, If given x_L, x_u and E_a we can find number of required iteration (n)

$$n = \log_2\left(\frac{\Delta x^0}{E_a}\right) \qquad \Delta x^0 = x_u - x_L$$

Example $f(x) = x e^{-x} - 2$

$x_L = 0$, $x_u = 1$ and $E_a = 0.022$

Find required # of iteration to reach $E_a = 0.022$

Solution

$$n = \log_2\left(\frac{\Delta x^0}{E_a}\right) = \log_2\left(\frac{1-0}{0.022}\right) = \log_2\left(\frac{1}{0.022}\right)$$

$n = 5.5 \Rightarrow n = 6$ iterations

not 5

Example $f(x) = x e^{-x} - 2$ $\qquad x_L = 0$, $x_u = 1$

Find E_a after $n=3$ iterations

$$n = \log_2\left(\frac{\Delta x^0}{E_a}\right) , \quad 2^{()} = 2^{()}$$

$$\Rightarrow E_a^{(n)} = \frac{\Delta x^0}{2^n}$$

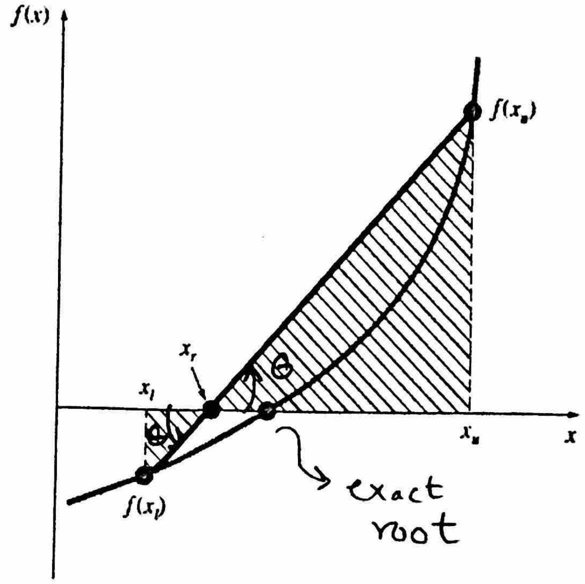
$$E_a^{(3)} = \frac{1-0}{2^3} \Rightarrow \boxed{E_a = 0.125}$$

$$E_a^{(6)} = \frac{1}{2^6} \Rightarrow \boxed{E_a = 0.015625}$$

Chapter 05

- False position Method (FPM)

FPM is very Similar to Bisection method but faster and little more accurate. However, little more complicated in theory.



- If function $f(x)$ has a root (x_r) lies in the interval $[x_L, x_u]$ and $f(x_L) \cdot f(x_u) < 0$, then FPM calculates x_r, as:

$$x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}$$

$$\tan\theta = \frac{f(x_u)}{x_u - x_r} = \frac{-f(x_L)}{x_r - x_L}$$

$$\Rightarrow x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}$$

Solution Procedure

① Check Sign $f(x_L) \cdot f(x_u) < 0$, If not, we cannot use FPM "No Solution"

② $x_r = x_u - \dfrac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}$

③ Find $f(x_r)$ and check Sign of $f(x_L) \cdot f(x_r)$

- If $f(x_L) \cdot f(x_r) = 0$, x_r = exact root
- If $f(x_L) \cdot f(x_r) > 0$ (+ve), $x_L = x_r$ and back to step 2
- If $f(x_L) \cdot f(x_r) < 0$ (−ve), $x_u = x_r$ and back to Step 2

$\Rightarrow$ Also, we need iterations in FPM

Example $f(x) = e^{x} - 2$ $\quad x_L = 0,\ x_u = 1$ Use FPM to find the root (x_r) for $\epsilon_a < 5\%$

Solution

Iteration 1

- Check $f(x_L) \cdot f(x_u) < 0$ $\quad (-)(+) = (-)$

$f(x_L) = f(0) = -1, \quad f(x_u) = f(1) = 0.7183 \Rightarrow f(x_L) \cdot f(x_u) < 0$ ✓

- $$x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}$$

$$= 1 - \frac{f(1)(0 - 1)}{f(0) - f(1)} = 1 - \frac{0.7183(-1)}{-1 - 0.7183} = 0.5819$$

$f(x_L) = f(0) = e^{0} - 2 = -1$

$f(x_u) = f(1) = e^{1} - 2 = 0.7183$

- $f(x_r) = f(0.5819) = e^{0.5819} - 2 = -0.2106$

$f(x_L) \cdot f(x_r) = (-ve)(-ve) = (+ve)$ $\quad x_L = x_r = 0.5819$

Iteration 2

$x_L = 0.5819, \quad x_u = 1$

$$x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)} = 1 - \frac{f(1)(0.5819 - 1)}{f(0.5819) - f(1)}$$

$$= 1 - \frac{0.7183(0.5819 - 1)}{-0.2106 - 0.7183} = 0.6770$$

$f(0.5819) = -0.2106$

$f(x_r) = f(0.677) = e^{0.677} - 2 = -0.032$

$f(x_L) \cdot f(x_r) = (-)(-) = +ve$ $\quad x_L = x_r = 0.6770$

Iteration #	x_L	x_u	x_r	Sign $f(x_L) \cdot f(x_r)$	ϵ_a
1	0	1	0.5819	(+)	N/A
2	0.5819	1	0.6770	(+)	$\epsilon_a = \left\| \frac{0.677 - 0.5819}{0.6770} \right\| 100\% = 14\%$
3	0.6770	1	0.6879	+	$\epsilon_a = \left\| \frac{0.6879 - 0.677}{0.6879} \right\| 100\% = 1.6\%$

$$\epsilon_a = \left| \frac{\text{Present } x_r - \text{Previous } x_r}{\text{present } x_r} \right| 100\%$$

$x_r = 0.6879$

Iteration 3

$x_L = 0.6770$, $x_u = 1$

$$x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)} = 1 - \frac{f(1)(0.677 - 1)}{f(0.677) - f(1)} = 0.6879$$

$f(0.677) = -0.032$

$f(1) = +0.7183$

Quick Comparison between Bisection and FPM

	# iterations	ϵ_a
Bisection	5	4.3 %
FPM	3	1.6 %

Faster
Accurate

needs more Calculations in x_r.