* Laplace Equation $\nabla^2 u = 0$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{y^2} \Rightarrow u_{xx} + u_{yy} = 0$ ①

$u_{xx} + u_{yy} = 0$ please note that Laplace equation is a simplied form of heat equation with steady state problem ,

$u_t = \alpha^2 (u_{xx} + u_{yy})$, S.S $\Rightarrow u_t = 0 \Rightarrow \boxed{u_{xx} + u_{yy} = 0}$

To solve laplace eq'n $\Rightarrow$ Separation of variables: $u(x,y) = X(x)\, Y(y)$

Subst. in PDE $\Rightarrow$ $\frac{X''}{X} = \frac{-Y''}{Y} = \text{constant} = -\lambda^2$ (we can $+\lambda^2$)

$X'' + \lambda^2 X = 0 \Rightarrow X(x) = A \cos \lambda x + B \sin \lambda x$

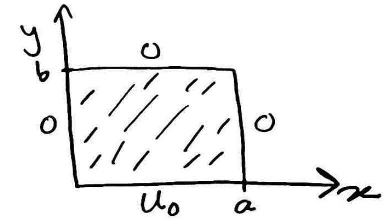
$Y'' - \lambda^2 Y = 0 \Rightarrow Y(y) = C \cosh \lambda y + D \sinh \lambda y$

A, B, C and D constants $\Rightarrow$ 4 BC's

Example: Solve $u_{xx} + u_{yy} = 0$

BC's : $u(0,y) = u(a,y) = 0$

$u(x,0) = u_0$, $u(x,b) = 0$



Sol. SoV: $u(x,y) = X(x)\, Y(y)$, subst. PDE

$\Rightarrow X''(x) + \lambda^2 X = 0 \Rightarrow X(x) = A \cos \lambda x + B \sin \lambda x$

$\Rightarrow Y''(y) - \lambda^2 Y = 0 \Rightarrow Y(y) = C \cosh \lambda y + D \sinh \lambda y$

Apply BC's

① $u(0,y) = 0 \Rightarrow X(0)\, Y(y) = 0 \Rightarrow X(0) = 0 = A + 0 \Rightarrow \boxed{A = 0}$, $X(x) = B \sin \lambda x$

② $u(a,y) = 0 \Rightarrow X(a)\, Y(y) = 0 \Rightarrow X(a) = 0 = \underset{\neq}{B}\, \underbrace{\sin \lambda a}_{0} \Rightarrow \lambda_n = \frac{n\pi}{a}$

Eigen value

The Eigen functions

$X_n(x) = \sin \frac{n\pi}{a} x$, $B_n = 1$

Thus

$Y_n(y) = C_n \cosh \lambda_n y + D_n \sinh \lambda_n y$, $\lambda_n = \frac{n\pi}{a}$

Apply BC's $u(x,b) = 0 = X(x)\, Y(b) \Rightarrow Y(b) = 0$

$Y_n(b) = 0 = C_n \cosh \lambda_n b + D_n \sinh \lambda_n b \Rightarrow C_n = -D_n \frac{\sinh(\lambda_n b)}{\cosh(\lambda_n b)}$

$\Rightarrow Y_n(y) = D_n \left[-\frac{\sinh \lambda_n b}{\cosh \lambda_n b} \cdot \cosh \lambda_n y + \sinh \lambda_n y \right]$

$$\Rightarrow Y_n(y) = \frac{-D_n}{\cosh\lambda_n b}\left[\sinh\lambda_n b \cosh\lambda_n y - \cosh\lambda_n b \sinh\lambda_n y\right]$$

$\alpha = \lambda_n b$

Remember ID: $\sinh\alpha\cosh\beta - \cosh\alpha\cdot\sinh\beta = \sinh(\alpha-\beta)$ $\quad \beta = \lambda_n y$

$$\Rightarrow Y_n(y) = \underbrace{\frac{-D_n}{\cosh\lambda_n b}}_{=E_n}\cdot\left(\sinh(\lambda_n(b-y))\right)$$

$$Y_n(y) = E_n \cdot \sinh(\lambda_n(b-y)) \quad , \quad E_n = \frac{-D_n}{\cosh\lambda_n b}$$

The PDE Sol. $u(x,y) = XY$

$$u_n(x,y) = \sum_{n=1}^{\infty} E_n\cdot\sinh\lambda_n(b-y)\cdot\sin\lambda_n x \quad , \quad \lambda_n = \frac{n\pi}{a}$$

Apply 2nd BC $\quad u_n(x,0) = u_0 = \sum_{n=1}^{\infty}\underbrace{E_n\cdot\sinh\lambda_n b}_{\text{constant } A_n}\cdot\sin(\lambda_n x)$

$$\Rightarrow u_n(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x = u_0$$

$$A_n = \frac{\langle u_0, \sin\frac{n\pi}{a}x\rangle}{\langle \sin\frac{n\pi}{a}x, \sin\frac{n\pi}{a}x\rangle} = \frac{4}{n\pi}u_0 \qquad n = \text{odd}$$

$$E_n = \frac{4u_0}{n\pi\sinh\lambda_n b}$$

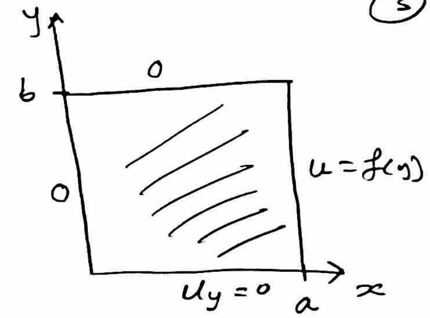
$$\Rightarrow u_n(x,y) = \sum_{n=1}^{\infty}\frac{4}{n\pi}u_0\frac{1}{\sinh\lambda_n b}\cdot\sinh(\lambda_n(b-y))\cdot\sin\lambda_n x \quad , \quad \lambda_n = \frac{n\pi}{a}$$

#

Example Solve $u_{xx} + u_{yy} = 0$

BC's $u(0,y) = 0 \qquad u(a,y) = f(y)$

$u_y(x,0) = 0 \qquad u(x,b) = 0$



Sol'n SOV: $u(x,y) = X(x)Y(y)$ subst. in PDE

$\Rightarrow X''Y + XY'' = 0 \div XY \Rightarrow \dfrac{X''}{X} = \dfrac{-Y''}{Y} = +\lambda^2$

$\Rightarrow Y'' + \lambda^2 Y = 0 \Rightarrow Y(y) = A\cos\lambda y + B\sin\lambda y$

$\Rightarrow X'' - \lambda^2 X = 0 \Rightarrow X(x) = C\cosh\lambda x + D\sinh\lambda x$

Apply BC's $u_y(x,0) = 0 \Rightarrow X(x)Y'(0) = 0 \Rightarrow Y'(0) = 0$

$Y'(y) = -Ay\sin\lambda y + \lambda B\cos\lambda y \Rightarrow Y'(0) = 0 = 0 + B \Rightarrow \boxed{B = 0}$

$Y(y) = A\sin\lambda y$

$Y(b) = 0 = A\cos\lambda b$, For non-trivial sol. $\Rightarrow \cos\lambda b = 0$

$\lambda b = \dfrac{(2n-1)\pi}{2} \Rightarrow \lambda_n = \dfrac{(2n-1)\pi}{2b}$, $n = 1, 2, \ldots$ "Eigen Values"

Eigen functions $Y_n(y) = \cos\lambda_n y$, $\lambda_n = \dfrac{(2n-1)\pi}{2b}$, $A_n = 1$ or any val.

$u(0,y) = 0 = X(0)Y(y) \Rightarrow X(0) = 0$, $X(0) = 0 = C + 0 = \boxed{C = 0}$

$\Rightarrow X_n(x) = D_n\sinh\lambda_n x$

$\Rightarrow u_n(x,y) = \sum_{n=1}^{\infty} X_n Y_n = \sum_{n=1}^{\infty} D_n\sinh\lambda_n x \cdot \cos\lambda_n y$, $\lambda_n = \dfrac{(2n-1)\pi}{2b}$

$u(a,y) = f(y) = \sum_{n=1}^{\infty} \underbrace{D_n\sinh\lambda_n a}_{= F_n} \cdot \cos\lambda_n y$ "Fourier series"

$\Rightarrow F_n = \dfrac{\langle F_n, \cos\lambda_n y\rangle}{\langle \cos\lambda_n y, \cos\lambda_n y\rangle} \Rightarrow D_n = \dfrac{F_n}{\sinh\lambda_n a}$

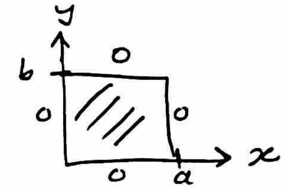
$\Rightarrow u_n(x,y) = \sum_{n=1}^{\infty} \dfrac{F_n}{\sinh\lambda_n a} \cdot \sinh\lambda_n x \cdot \cos\lambda_n y$

Non-homogeneous elliptic PDE

$\nabla^2 u = f(x,y) \Rightarrow u_{xx} + u_{yy} = f(x,y)$, we'll focus only on $F(x,y) = F$ "constant"

Example Solve $u_{xx} + u_{yy} = F$ ← constant

BC's $u(0,y) = u(a,y) = 0$
$u(x,0) = u(x,b) = 0$



Sol'n : we need to do change of variables

let $u(x,y) = v(x,y) + w(x)$, or $w(y)$

$\Rightarrow u_{xx} = v_{xx} + w''$ and $u_{yy} = v_{yy} + 0$ ← Subst. in PDE.

$\Rightarrow v_{xx} + v_{yy} + w''(x) = F$ → $v_{xx} + v_{yy} = 0$, $w''(x) = F$ } Principle of superposition!

$v_{xx} + v_{yy} = 0$

apply BC's

① $u(0,y) = \underbrace{v(0,y)}_{=0} + \underbrace{w(0)}_{=0} = 0 \Rightarrow w(0) = 0$ and $v(0,y) = 0$

② $u(a,y) = v(a,y) + w(a) = 0 \Rightarrow w(a) = 0$ and $v(a,y) = 0$

③ $u(x,0) = 0 = v(x,0) + w(x) \Rightarrow v(x,0) = -w(x)$

④ $u(x,b) = 0 = v(x,b) + w(x) \Rightarrow v(x,b) = -w(x)$

Therefore: PDE

$v_{xx} + v_{yy} = 0$

BC's $v(0,y) = v(a,y) = 0$
$v(x,0) = v(x,b) = -w(x)$

ODE $\Rightarrow w''(x) = F$
$w(0) = 0$
$w(a) = 0$

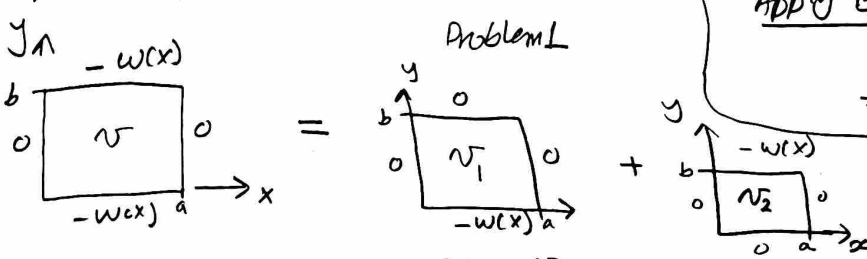
$\Rightarrow w'(x) = Fx + A$ ← constants

$w(x) = \frac{Fx^2}{2} + Ax + B$

Apply BC's $w(0) = 0 \Rightarrow \boxed{B = 0}$
$w(a) = 0 \Rightarrow A = -\frac{F.a}{2}$

$\Rightarrow w(x) = \frac{Fx}{2}(x - a)$

This PDE can be splitted into 2 PDE's



$v_{1xx} + v_{1yy} = 0$ $v_{2xx} + v_{2yy} = 0$

Now, Problem 1 and Problem 2 can be solved seperately "as we did before"

and find $v_1(x,y)$ and $v_2(x,y) \Rightarrow v(x,y) = v_1 + v_2$ "superposition Principle"

Finally, $u(x,y) = v_1(x,y) + v_2(x) + w(x)$

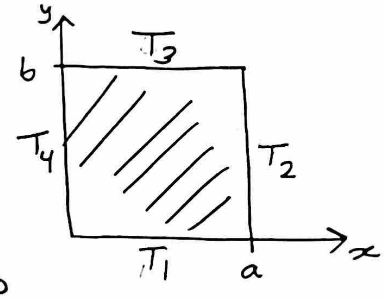
$v_1(x,y) = \sum_{n=1}^{\infty} B_n \sinh \lambda_n(b-y) \sin \lambda_n x \cdot A_n$ $B_n = \frac{\langle -w(x), \sin \lambda_n x \rangle}{\langle \sin \lambda_n x, \sin \lambda_n x \rangle}$, $\lambda_n = \frac{n\pi}{a}$

$v_2(x,y) = \sum_{n=1}^{\infty} B_n \sinh \lambda_n y \cdot \sin \lambda_n x \cdot A_n$ $A_n = \sinh \lambda_n b$

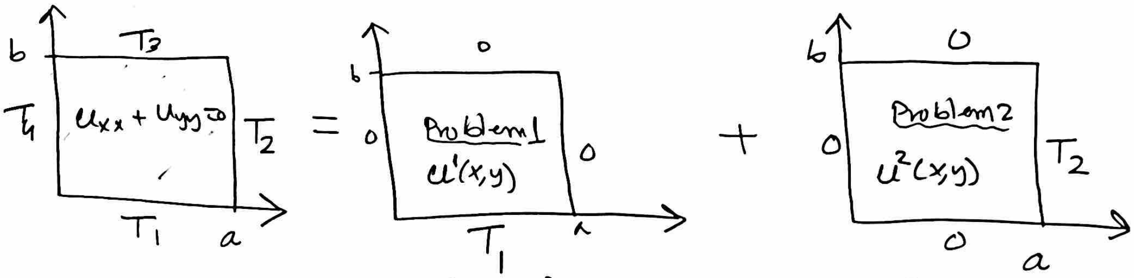
Example: Solve $u_{xx} + u_{yy} = 0$

BC's $u(0,y) = T_4$, $u(x,0) = T_1$

$u(a,y) = T_2$, $u(x,b) = T_3$



Sol'n: This problem can be splitted into 4 problems like



$u^1_{xx} + u^1_{yy} = 0$

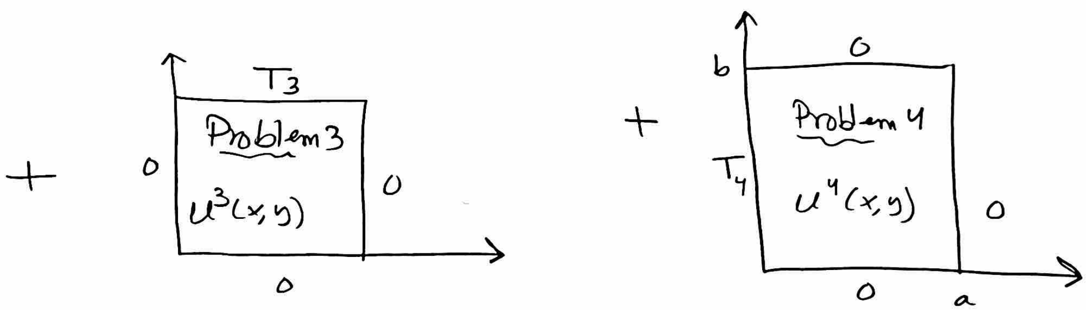
$u^1(x,0) = T_1$

other BC's $= 0$

$u^2_{xx} + u^2_{yy} = 0$

$u^2(a,y) = 0$

all other BC's $= 0$



$u^3_{xx} + u^3_{yy} = 0$

$u^3(x,b) = T_3$

all other BC's $= 0$

$u^4_{xx} + u^4_{yy} = 0$

$u^4(0,y) = T_4$

all other BC's $= 0$

$\Rightarrow$ we can obtain u^1, u^2, u^3 and u^4

then

$u(x,y) = u^1 + u^2 + u^3 + u^4$

"Principle of superposition!"