12.5 and 12.6 : Heat Equation ①

For 3D Heat-equation : $u_t = \alpha^2 \nabla^2 u$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

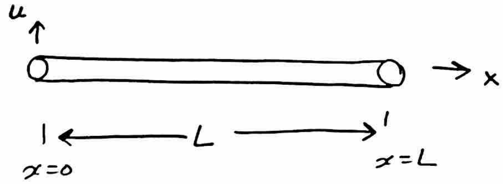
thermal diffusivity (α^2) — Laplace operator (∇^2)

If $u(x,y,z,t) \Rightarrow u_t = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$

For this equation, we need 6 BC's and 1 IC .

For 1-D Heat equation $u_t = \alpha^2 u_{xx}$, $u(x,t)$

here, we need 2 BC's and 1 IC.



Homogenous BC's

$u(0,t) = 0$, $u(L,t) = 0$

IC $u(x,t) = \phi(x)$

For 2-D Heat equation $u(x,y,t)$

$$u_t = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \alpha^2 (u_{xx} + u_{yy})$$

For 2D, we mainly aim to solve steady-state solution $u_t = 0$

The equation becomes :

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \Rightarrow \text{Commonly Known : Laplace eq'n.}$$

For this problem, we need 4 BC's and No IC's.

Let's focus on 1-D $\quad u_t = \alpha^2 u_{xx}$ $\quad u(0,t) = u(L,t) = 0$

$u(x,0) = \phi(x)$

To solve this PDE, we use separation of variables

$u(x,t) = X(x)\,T(t)$, Subst. in PDE

$$\Rightarrow \quad XT' = \alpha^2 X''(x)\,T(t) \qquad \div \; XT$$

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X(x)} = \text{constant} = -\lambda^2$$

For $T(t)$ $\quad T' + \alpha^2\lambda^2 T = 0$ $\quad$ "1st order ODE" $\quad T(t) = C e^{-\alpha^2\lambda^2 t}$

For $X(x)$ $\quad X'' + \lambda^2 X = 0$ $\quad$ "2nd order ODE" $\quad X(x) = A\cos\lambda x + B\sin\lambda x$

$$\Rightarrow \quad u(x,t) = C e^{-\alpha^2\lambda^2 t}\left(A\cos\lambda x + B\sin\lambda x\right)$$

To find A and B, apply BC's

1. $u(0,t) = 0 = \cancel{C e^{-\alpha^2\lambda^2 t}}\,(A + 0) = 0 \Rightarrow A = 0$

$$u(x,t) = C e^{-\alpha^2\lambda^2 t} \cdot B\sin\lambda x$$

2. $u(L,t) = 0 = \cancel{C e^{-\alpha^2\lambda^2 t}} \cdot B\sin\lambda L = 0$

$\Rightarrow B\sin\lambda L = 0$ $\quad \Rightarrow$ For non-trivial sol'n (EVP)

B=0 or $\quad$ sin = 0

trivial sol'n

$\sin\lambda L = 0$

$\lambda L = n\pi \Rightarrow \lambda_n = \dfrac{n\pi}{L}$

Eigenvalues.

Therefore

$X_n(x) = B_n \sin\lambda_n x \Rightarrow$ Eigenfunctions!

then

$$U_n(x,t) = C e^{-\alpha^2\lambda_n^2 t} \cdot B_n \sin\lambda_n x$$

$$U_n(x,t) = b_n \sin\frac{n\pi}{L}x \cdot e^{-\alpha^2 \frac{n^2\pi^2}{L^2} t}, \quad b_n = C \cdot B_n$$

To find b_n, we apply IC

$$u(x,0) = \phi(x) \Rightarrow U_n(x,0) = b_n \sin \lambda_n x = \phi_n$$

Using Fourier series

$$b_n = \frac{\langle \phi_n , \sin \frac{n\pi}{L}x \rangle}{\langle \sin \frac{n\pi}{L}x, \sin \frac{n\pi}{L}x \rangle} = \frac{\int_0^L \phi_n(x) \sin \frac{n\pi}{L}x \, dx}{\int_0^L \sin \frac{n\pi}{L}x \cdot \sin \frac{n\pi}{L}x \, dx} = \frac{2}{L} \int_0^L \phi_n \cdot \sin \frac{n\pi}{L} dx$$

Finally, the Solution is written in Form of infinite series

$$U_n(x,t) = \sum_{n=1}^{\infty} b_n \cdot \sin \frac{n\pi}{L}x \cdot e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$$

$$b_n = \frac{2}{L} \int_0^L \phi_n \cdot \sin \frac{n\pi}{L}x \, dx \; !$$

Other types of BC's "Non-homog."

1. Dirichlet BC's

$u(0,t) = g_1(t)$ and $u(L,t) = g_2(t)$, If $u(0,t) = u(L,t) = 0$ $\Rightarrow$ Homog.

or $u(0,t) = u_1$ and $u(L,t) = u_2$

2. Neumann's conditions

$u_x(0,t) = Q_1(t)$, $u_x(L,t) = Q_2(t)$, $u_x = \frac{\partial u}{\partial x}$

3. Robin's BC's or Mixed BC's

$u_x(0,t) + b_1 u(0,t) = h_1(t)$

$u_x(L,t) + b_2 u(L,t) = h_2(t)$

b_1 and b_2 Constants

Example Solve $u_t = \alpha^2 u_{xx}$ $0 \le x \le L$

$u(0,t) = u_1$, $u(L,t) = u_2$, $u(x,0) = \phi(x)$

Solution: - we first need to transform the BC's to a homogeneous form. Let $u(x,t) = v(x,t) + V(x)$ ← Subst in PDE

$u_t = v_t + 0 = v_t$, $u_{xx} = v_{xx} + V_{xx}$

to reduce $u_{xx} = v_{xx}$, let $V_{xx} = 0 \Rightarrow V(x) = C + Dx$

$\Rightarrow v_t = \alpha^2 (v_{xx}) \Rightarrow$ PDE

- Now, apply BC's in $u(x,t) = v(x,t) + V(x)$

① $u(0,t) = u_1 = v(0,t) + V(0)$. To have homog. BC's $\boxed{v(0,t) = 0}$

and then $\boxed{V(0) = u_1}$

② $u(L,t) = u_2 = v(L,t) + V(L)$. To have homog. BC's $\boxed{v(L,t) = 0}$

and then $\boxed{V(L) = u_2}$

Now, the PDE

$v_t = \alpha^2 v_{xx} \rightarrow$ $v(0,t) = 0$, $v(L,t) = 0$, $v(x,0) = ?$

$V(x) = C + Dx$ → $V(0) = u_1$, $V(L) = u_2$

$V(0) = \boxed{u_1 = C}$

$V(L) = u_2 = C + DL \Rightarrow D = \frac{u_2 - u_1}{L}$

$\Rightarrow V(x) = u_1 + \frac{u_2 - u_1}{L} x$

$u(x,0) = v(x,0) + V(x) = \phi(x)$

$\Rightarrow v(x,0) = \phi(x) - \left(u_1 + \frac{u_2 - u_1}{L} x\right)$

$v(x,0) = f(x)$

So: $v_t = \alpha^2 v_{xx}$ $v(0,t) = 0$, $v(L,t) = 0$, $v(x,0) = f(x)$

From previous solution

$$v_n(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cdot e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \quad , \quad b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x \, dx$$

Remember $u(x,t) = v(x,t) + V(x)$

$$\Rightarrow u_n(x,t) = \left(\sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cdot e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \right) - \left(u_1 + \frac{u_2 - u_1}{L} x\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \quad !$$

Example: solve $u_t = \alpha^2 u_{xx}$ $0 \leq x < L$

$u(0,t) = u_x(L,t) = 0$ & $u(t,0) = \phi(x)$

Sol'n: Separation of variables $u(x,t) = X(x)\,T(t)$ sub in PDE.

$$\Rightarrow XT' = \alpha^2 X'' T \Rightarrow \frac{T'}{\alpha^2 T} = \frac{X''}{X} = -\lambda^2 = \text{const}$$

$$\Rightarrow T' + \alpha^2\lambda^2 T = 0 \rightarrow 1^{st} \text{ order ODE} \Rightarrow T(t) = C e^{-\alpha^2\lambda^2 t}$$

$$X'' + \lambda^2 X = 0 \Rightarrow 2\text{nd order ODE} \Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x$$

To find A & B $\Rightarrow$ apply BC's

① $u(0,t) = 0 = X(0)T(t) \Rightarrow X(x) = 0 = A + 0 \Rightarrow \boxed{A = 0}$

$\Rightarrow X(x) = B\sin\lambda x$, $X'(x) = \lambda B\cos\lambda x$

② $u_x(0,t) = 0 \Rightarrow \lambda B \cos\lambda L = 0$ $\lambda = 0$ or $B = 0$ Trivial Sol'n   or $\cos\lambda L = 0$ non-trivial sol

$\Rightarrow \cos\lambda L = 0 \Rightarrow \boxed{\lambda = \frac{(2n-1)\pi}{2L}}$ Eigen values.

and $X_n(x) = B_n \sin\frac{(2n-1)}{2L}\pi x$ ← Eigen function

$$\Rightarrow u_n(x,t) = B_n \cdot \sin\lambda_n x \cdot C e^{-\alpha^2\lambda_n^2 t} \quad , \quad B_n \cdot C = b_n$$

$$= b_n \sin\lambda_n x \cdot e^{-\alpha^2\lambda_n^2 t}$$

To find b_n, apply IC's $u(0,t) = \phi(x)$

$$u_n(0,t) = \phi_n(x) = b_n \sin\lambda_n x$$

Using Fourier serie

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\lambda_n x \cdot e^{-\alpha^2\lambda_n^2 t} \quad , \quad \lambda_n = \frac{(2n-1)\pi}{2L}$$

$$b_n = \frac{\langle \phi_n(x), \sin\lambda_n x \rangle}{\langle \sin\lambda_n x, \sin\lambda_n x \rangle}$$