Chapter 12 Partial Differential Equations (PDE's)

12.1

Many Engineering Problems, are commonly described by PDE's:

* General form of 2nd order linear PDE's of two independent variables:

$$A\,u_{xx} + B\,u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

where: $\frac{\partial^2 u}{\partial x^2} = u_{xx}$, $\frac{\partial^2 u}{\partial y^2} = u_{yy}$, $\frac{\partial^2 u}{\partial x \partial y} = u_{xy}$

$\frac{\partial u}{\partial x} = u_x$, $\frac{\partial u}{\partial y} = u_y$, $u = u(x,y)$

and A, B, C, D, E, F and G are coefficients and they can be both functions of x, y and/or constants

* This type of PDE's can be classified into 3 types based on A, B, and C "The 2nd derivative coff."

① Hyperbolic Equation $B^2 - 4AC > 0$

↳ Example: Wave equation $u_{tt} = c^2 u_{xx}$

$\Rightarrow c^2 u_{xx} - u_{tt} = 0$ $\quad B = 0,\ A = c^2,\ C = -1$

$B^2 - 4AC \Rightarrow (0)^2 - 4(c^2)(-1) = 4c^2 > 0$

This equation can be both IVP and BVP and EVP

② Parabolic Equation $B^2 - 4AC = 0$

↳ Example: Heat Equation $u_t = c^2 u_{xx}$

$\Rightarrow c^2 u_{xx} - u_t = 0 \Rightarrow B = 0,\ A = c^2,\ C = 0$

$B^2 - 4AC = 0 - 4(c^2)(0) = 0$ ← Parabolic equation

This eq'n can be both IVP and BVP

③ Elliptic Equation $B^2 - 4AC < 0$

↳ Example: Laplace eq'n: $u_{xx} + u_{yy} = 0$, $B = 0,\ A = 1,\ C = 1$

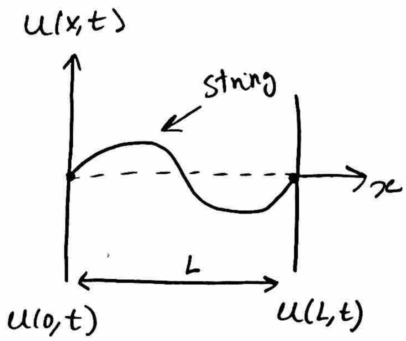
$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$

This eq'n is BVP only!

12.2,3,4) Wave Equation

1-D wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ or $u_{tt} = c^2 u_{xx}$

$u(x,t)$ and $c^2 = \frac{T}{\rho}$ T: Tension in string, ρ: density



we need two initial conditions u_{tt} and Two Boundary conditions u_{xx}

12.2

* Generally, to Solve PDE's, we use "Separation of variables"

$u(x,t) = X(x)T(t)$ ← Subst in PDE $u_{tt} = XT''$, $u_{xx} = X''T$

$\Rightarrow XT'' = c^2 X''T \Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = \text{constant} = -\lambda$

$\Rightarrow X'' + \lambda^2 X = 0$
$T'' + c^2\lambda^2 T = 0$ } Both are 2nd order ODE's. To Solve we use characterisitic eq'n.

Therefore, $X(x) = A\cos\lambda x + B\sin\lambda x$
$T(t) = C\cos c\lambda t + D\sin c\lambda t$

If the boundary conditions are $u(0,t)=0$ and $u(L,t)=0$

$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0 \Rightarrow A + 0 = 0 \Rightarrow \boxed{A=0}$

therefore: $X(x) = B\sin\lambda x$

$u(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0 \Rightarrow B\sin\lambda L = 0$ $B \neq 0 \Rightarrow \lambda_n = \frac{n\pi}{L}$ Eigen values

$X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$ Eigen functions!

$u_n(x,t) = \sin\lambda_n x\left(C_n\cos c\lambda_n t + D_n\sin c\lambda_n t\right)$

Using Fourier Series

$$U_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x\left(C_n \cos\frac{n\pi}{L}ct + D_n \sin\frac{n\pi}{L}ct\right)$$

If the initial conditions are $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

To apply the initial conditions, we need to find $U_{nt} = \frac{\partial u_n}{\partial t}$

$$U_{nt} = \frac{\partial u_n}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x\left[-C_n \frac{n\pi}{L}c \sin\frac{n\pi}{L}ct + D_n \frac{n\pi}{L}c \cos\frac{n\pi}{L}ct\right]$$

$$u_n(x,0) = f(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x \cdot C_n \Rightarrow C_n = \frac{\langle f(x), \sin\frac{n\pi}{L}x\rangle}{\langle \sin\frac{n\pi}{L}x, \sin\frac{n\pi}{L}x\rangle}$$

$$U_{nt}(x,0) = g(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x \cdot D_n \cdot \frac{n\pi}{L}c \Rightarrow D_n = \frac{\langle g(x), \frac{n\pi}{L}c \sin\frac{n\pi}{L}x\rangle}{\langle \frac{n\pi}{L}c \sin\frac{n\pi}{L}x, \frac{n\pi}{L}c \sin\frac{n\pi}{L}x\rangle}$$

In vibration, λ_n are natural frequencies, $X_n(x)$ are modeshapes and $U_n(x,t)$ is the vibration response!

Example $\quad u_{tt} = c^2 u_{xx}$ $\quad u(0,t) = u(x,t) = 0$

$u(x,0) = f(x)$, $u_t(x,0) = 0$

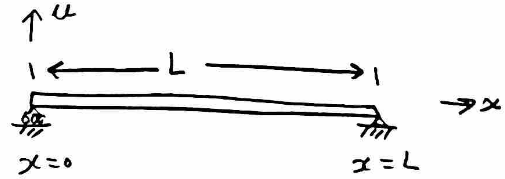
Sol'n : From the previous derivation $g(x) = 0$ thus $D_n = 0$

Therefore: $U_n(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L}x \cdot \cos\frac{n\pi}{L}ct$

$$C_n = \frac{\langle f(x), \sin\frac{n\pi}{L}x\rangle}{\langle \sin\frac{n\pi}{L}x, \sin\frac{n\pi}{L}x\rangle}$$

* Vibration of Simply-supported Beam - Prob 16 page 552

$$u_{tt} = -\frac{EI}{\rho A} u_{xxxx} \qquad u_{xxxx} = \frac{\partial^4 u}{\partial x^2}$$



BC's
$u(0,t) = u(L,t) = 0$
$u_{xx}(0,t) = u_{xx}(L,t) = 0$

IC's
$u(x,0) = f(x)$
$u_t(x,0) = g(x)$

E: Young's Mod.
I: Moment of Inertia
ρ: Density
A: Cross-sectional area

Solution - Use Sep of var.

$u(x,t) = X(x)\,T(t)$ sub in PDE

$$\frac{T''}{\alpha^2 T} = \frac{-X^{(4)}}{X} = -\lambda^2 \qquad \text{where } \alpha^2 = \frac{EI}{\rho A}$$

$T'' + \alpha^2\lambda^2 T = 0 \Rightarrow T(t) = A\cos\alpha\lambda + B\sin\alpha\lambda t$ $\qquad \hat{i} = \sqrt{-1}$

$X^{(4)} - \lambda^2 X = 0 \Rightarrow$ Charact. eq'n $r^4 - \lambda^2 = 0 \Rightarrow r_{1,2,3,4} = \pm\sqrt{\lambda} \pm \hat{i}\sqrt{\lambda}$

$\Rightarrow X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x + E\cosh(\lambda x) + F\sinh(\sqrt{\lambda}x)$

To find C, D, E and F $\Rightarrow$ Apply BC's
$\Rightarrow X''(x) = \lambda(\bar{C}\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x + E\cosh\sqrt{\lambda}x + F\sinh\sqrt{\lambda}x$

$$\left.\begin{array}{l} X(0) = 0 = C + E \\ X''(0) = 0 = -C + E \end{array}\right\} C = E = 0 \Rightarrow X(x) = (D\sin\sqrt{\lambda}x + F\sinh\sqrt{\lambda}x)$$

$X'' = \lambda(D\sin + F\sinh)$

$X(L) = 0 \Rightarrow (D\sin\sqrt{\lambda}L + F\sinh\sqrt{\lambda}L) = 0 \quad -(a)$

$X''(L) = 0 \Rightarrow \lambda(D\sin\sqrt{\lambda}L + F\sinh\sqrt{\lambda}L) = 0 \quad -(b)$

If we take $(a+b) \Rightarrow \underbrace{2F}_{=0}$ or $\underbrace{\sinh\sqrt{\lambda}L}_{=0} = 0$

$\sinh\sqrt{\lambda}L = 0$ only if $\lambda = 0$
" trivial Sol."
$\Rightarrow F = 0$

$\Rightarrow X(x) = D\sin\sqrt{\lambda}L$

$$\left.\begin{array}{l} X(L) = 0 = D\sin\sqrt{\lambda}L \\ X''(L) = 0 = D\sin\sqrt{\lambda}L \end{array}\right\} \text{Same eq} \Rightarrow \underbrace{D}_{=0} \text{ or } \underbrace{\sin\sqrt{\lambda}L}_{=0} = 0$$

$\cancel{D=0}$ trivial sol'n $\Rightarrow \sin\sqrt{\lambda}L = 0$

$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}$ Eigen values

$n = 1, 2, \ldots$

$X_n(x) = D_n \sin\sqrt{\lambda_n} = D_n \sin\frac{n\pi}{L}x$ ← Eigenfunctions.

D_n can be any value, let $D_n = 1$

$\boxed{X_n(x) = \sin\frac{n\pi}{L}x}$ Eigen fun. or modeshapes!

$$u_n(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L}x \left(A_n \cos \frac{n\pi}{L}\alpha t + B_n \sin \frac{n\pi}{L}\alpha t \right) \quad , \alpha = \sqrt{\frac{EI}{\rho A}}$$

We can now apply IC's, as we did Before:

$$A_n = \frac{\langle f(x), \sin \frac{n\pi}{L}x \rangle}{\langle \sin \frac{n\pi}{L}x, \sin \frac{n\pi}{L}x \rangle} \quad , \quad B_n = \frac{\langle g(x), \frac{n\pi}{L}\alpha \sin \frac{n\pi}{L}x \rangle}{\langle \frac{n\pi}{L}\alpha \sin \frac{n\pi}{L}x, \frac{n\pi}{L}\alpha \sin \frac{n\pi}{L}x}$$

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practice Problem set 12.2 Problem 19 page 552.

Types of BC's in wave Equation

1. **Homogeneous BC's**

$u(0,t) = u_1$
$u(L,t) = u_2$

u_1 and u_2 constants

2. **Dirchilite's BC's** → Non-homog.

$u(0,t) = h_1(t)$
$u(L,t) = h_2(t)$

3. **Neumann's BC's** → non-homg.

$u_x(0,t) = h_1(t)$
$u_x(L,t) = h_2(t)$

4. **Mixed BC's** → Non-homog.

$u_x(0,t) = \frac{K}{T}\left(u(0,t) - h_1(t)\right)$

$u_x(L,t) = \frac{K}{T}\left(u(L,t) - h_2(t)\right)$

K, T constants

Example: Solve $u_{tt} = c^2 u_{xx}$, $u(0,t) = \epsilon \sin \overset{\text{omega}}{\omega} t$, $u(L,t) = 0$, $u(x,0) = u_t(x,0) = 0$

Sol'n: For this case $f(x) = g(x) = 0 \Rightarrow$ No sol! what should we do?

– we need transformation of variables.

let $\Rightarrow u(x,t) = v(x,t) + w(x,t)$

this will transform the PDE to be in terms of v

and BC's as: $v(0,t) = \epsilon \sin \omega t$, $v(L,t) = 0$

IC's as: $v(x,0) = 0$, $v_t(x,0) = g(x)$

Find u_{tt} and u_{xx} and subst. in PDE

$u_{tt} = v_{tt} + w_{tt}$, $u_{xx} = v_{xx} + w_{xx}$

$\Rightarrow$ PDE $\Rightarrow v_{tt} + w_{tt} = \alpha^2 v_{xx} + \alpha^2 w_{xx}$

① $v_{tt} = \alpha^2 v_{xx}$
② $w_{tt} = \alpha^2 w_{xx}$

$\Big\}$ Solve and find v and w then $u = v + w$

For $v \Rightarrow v_{tt} = \alpha^2 v_{xx}$, BC's $\Rightarrow v(0,t) = \epsilon \sin \omega t$, $v(L,t) = 0$

IC's $\Rightarrow v(x,0) = 0$, $v_t(x,0) = g(x)$

For $w \Rightarrow w_{tt} = \alpha^2 w_{xx}$, BC's $\Rightarrow w(0,t) = \overbrace{u(0,t)}^{\epsilon \sin \omega t} - \overbrace{v(0,t)}^{\epsilon \sin \omega t} = 0$

$w(L,t) = \underbrace{u(L,t)}_{0} - \underbrace{v(L,t)}_{0} = 0$

IC's $\Rightarrow w(x,0) = \overset{0}{u(x,0)} - \overset{0}{v(x,0)} = 0$

$w_t(x,0) = u_t(x,0) - v_t(x,0)$

$= 0 - g(x) = -g(x) = G(x)$

For $w(x,t) \Rightarrow$ we have solved it previously. Thus:

$$w_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \, \sin\frac{n\pi}{L}\alpha t$$

where $A_n = \dfrac{\langle G(x), \sin\frac{n\pi}{L}x \rangle}{\langle \sin\frac{n\pi}{L}x, \sin\frac{n\pi}{L}x \rangle}$, $G(x) = -g(x)$ could be any non-zero function

For $v(x,t) \Rightarrow v_{tt} = \alpha^2 v_{xx}$, use Sep of Var. to solve

$v(x,t) = X(x)T(t)$ Subst. in PDE $\Rightarrow XT'' = \alpha^2 X''T$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{\alpha^2 T} = -\lambda^2$$

$T'' + \alpha^2\lambda^2 T = 0$, $T(t) = A\cos\alpha\lambda t + B\sin\alpha\lambda t$

$X'' + \lambda^2 X = 0$, $X(x) = A_0\cos\lambda x + B_0 \sin\lambda x$

Apply BC's $\quad v(0,t) = \epsilon\sin\omega t$, $v(L,t) = 0$

1. $v(0,t) = \underbrace{\epsilon\sin\omega t}_{\text{only fun of time}} = X(0)T(t)$ $\Rightarrow X(0)=1$ $\Rightarrow T(t) = \epsilon\sin\omega t$

$T(t) = \epsilon\sin\omega t$, compare to $T(t) = A\cos\alpha\lambda t + B\sin\alpha\lambda t$

$\Rightarrow A = 0$, $\omega = \alpha\lambda$, $B = \epsilon$

2. $v(L,t) = 0 = \underbrace{X(L)}_{=0}\underbrace{T(t)}_{\neq 0} \Rightarrow X(L) = 0 = A_0\cos\lambda L + B_0\sin\lambda L$

From BC1 $\leftarrow X(0) = 1 = A_0 + 0 \Rightarrow A_0 = 1$

$\Rightarrow X(x) = \cos\lambda x + B_0\sin\lambda x$

$X(L) = \cos\lambda L + B_0\sin\lambda L = 0 \Rightarrow B_0 = \dfrac{-\cos\lambda L}{\sin\lambda L} = -\cot\lambda L$

$\Rightarrow X(x) = \cos\lambda x - \cot\lambda L \cdot \sin\lambda x \qquad (\lambda = \frac{\omega}{\alpha})$

$T(t) = \epsilon\sin\omega t$

$$\Rightarrow v(x,t) = X(x)T(t) = \epsilon\sin\omega t\left(\cos\frac{\omega}{\alpha}x - \cot\frac{\omega}{\alpha}L\cdot\sin\frac{\omega}{\alpha}x\right)$$

$$\Rightarrow u(x,t) = v(x,t) + w(x,t)$$

$$= \epsilon\sin\omega t\left(\cos\frac{\omega}{\alpha}x - \cot\frac{\omega}{\alpha}L\cdot\sin\frac{\omega}{\alpha}x\right) + \sum_{n=1}^{\infty} A_n\sin\frac{n\pi}{L}x\cdot\sin\frac{n\pi}{L}\alpha t$$

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