* Function spaces

For 3 Dimensional vectors $\vec{u}$ and $\vec{v}$, can be written as:

$$\vec{u} = [\underset{x}{u_1}, \underset{y}{u_2}, \underset{z}{u_3}] \quad \text{and} \quad \vec{v} = [\underset{x}{v_1}, \underset{y}{v_2}, \underset{z}{v_3}]$$

* The dot Product $<\vec{u}, \vec{v}> = u_1 v_1 + u_2 v_2 + u_3 v_3$ "scalar"

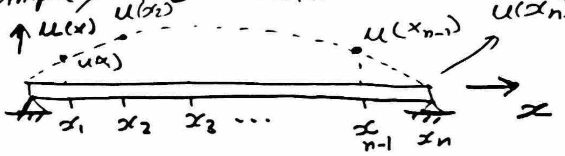
For n-Dimensional vectors

$\vec{u} = [u_1, u_2, \dots, u_n]$ and $\vec{v} = [v_1, v_2, \dots, v_n]$, thus:

$$<\vec{u}, \vec{v}> = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

* We can extend this definition for continuous functions $u(x)$ and $v(x)$ where $a \le x \le b$. For example, the beam

Approximately, $\vec{u} \cong [u(x_1), u(x_2), \dots, u(x_n)]$



For $n \to \infty$, the dot Product "inner product"

$$<u(x), v(x)> = \int_a^b u(x) v(x) dx$$

, If $<u, v> = 0$, then u and v are orthogonal!

Example If $u(x) = \sin(x)$, $v(x) = \cos(x)$ $\quad 0 \le x \le 2\pi$, Find $<u, v>$

Sol'n $<u, v> = \int_0^{2\pi} \sin x \cos x \, dx$, remember $\cos x \sin x = \frac{1}{2} \sin 2x$

$$= \int_0^{2\pi} \frac{\sin 2x}{2} dx = \left. \frac{-\cos 2x}{4} \right|_0^{2\pi} = \frac{-1}{4}(1-1) = 0$$

therefore, sin and cos functions are orthogonal!

Now, Consider set of functions: $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$, $a \le x \le b$

these functions are orthogonal if:

$$<\phi_m(x), \phi_n(x)> = \int_a^b \phi_m(x) \phi_n(x) dx = 0 \quad \text{for } m \ne n$$

and if $m = n$, they are orthogonal if =

$$<\phi_n(x), \phi_n(x)> = \int_0^1 [\phi_n(x)]^2 dx = 1$$

* Sturm - Liouville theory (SL - theory) "section 11.5"

Differential Eigen value problems (EVP) are expressed as:

$L\,y = \lambda y$, where L is linear differential operator , λ: real value

and $L = \frac{-1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right] \qquad a \le x \le b$

therefore: $[p(x)\,y']' + [q(x) + \lambda w(x)]\,y = 0 \qquad -(1)$

Now, the definition of the inner products can be extended as:

$$< f(x), g(x) > = \int_a^b f(x)\,g(x)\,w(x)\,dx \;, \quad w(x): \text{weight function}$$

The EVP problem above (Eq(1)), requires homogenous boundary conditions as:-

$\alpha\, y(a) + \beta\, y'(a) = 0$

$\gamma\, y(b) + \delta\, y'(b) = 0$

$\begin{cases} \alpha, \beta, \gamma \text{ and } \delta : \text{constants} \\ \text{where the pairs } (\alpha,\beta) \text{ and } (\gamma,\delta) \text{ cannot be both zero!} \end{cases}$

* Remember $(uv)' = uv' + vu'$

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b vu'\,dx$$ " integration by parts "

* The General Sturm - Louiville (SL) Problem " Eq(1), becomes"

$$p(x)\,y'' + p'(x)\,y' + [q(x) + \lambda w(x)]\,y = 0 \qquad - Eq(2)$$

- Properties of SL problem

① It has a sequence of eigenvalues
$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$

② For each λ_n "eigenvalue", there is a non-zero eigen function $\phi_n(x)$

③ the eigen functions corresponding to an eigenvalue are always orthogonal

$$< \phi_m(x), \phi_n(x) > = \int_a^b \phi_m(x)\,\phi_n(x)\,dx = 0 \leftarrow \text{orthogonal!}$$

Famous example of SL problem $\Rightarrow y'' + \lambda y = 0 \qquad \begin{cases} y(0) = 0 \\ y(L) = 0 \end{cases}$ $0 \le x \le L$
BC's.

If we compare to eq(2) $\quad p(x) = 1 \rightarrow p'(x) = 0$
$q(x) = 0$ and $w(x) = 1$

* Solution procedure for Solving SL Problems

- We normally need to deal with

$$\lambda < 0 \;,\; \lambda = 0 \;,\; \text{and} \;\; \lambda > 0$$

remember, characteristic equation Ch. 2

$r^2 = -\lambda$

$r = \pm\sqrt{\lambda}$

① If $\lambda = 0$ then the ode is $y''(x) = 0$ this will lead to trivial Sol'n.

② $\lambda < 0 \rightarrow$ The Solution will be in exponential terms ($e^{()}$) which will not satisfy the BC's

③ $\lambda > 0$ " This is the most common type " therefore, the soln $\Rightarrow y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

Back to the previous example: $y'' + \lambda y = 0$

$$\left.\begin{aligned} y(0) &= 0 \\ y(L) &= 0 \end{aligned}\right\} \rightarrow \text{Boundary Conditions}$$

$$y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \quad , \text{ to find } A \,\&\, B \rightarrow BC's$$

$y(0) = 0 = A \Rightarrow A = 0, \quad y(x) = B\sin\sqrt{\lambda}x$

$y(L) = 0 = B\sin\sqrt{\lambda}L = 0 \quad \begin{cases} B = 0 & (\text{trivial sol'n } y(x) = 0\text{"} \times \\ \text{or} & \\ \sin\sqrt{\lambda}L = 0 & \Rightarrow \sqrt{\lambda}L = n\pi \text{ or } \underbrace{\lambda_n = \frac{n^2\pi^2}{L^2}}_{\text{Eigenvalues}} \end{cases}$

therefore, the eigenfunctions

$$y_n(x) = \sin\left(\frac{n\pi}{L}x\right) \leftarrow \text{Eigen functions!}$$

Test the orthogonality condition:

$$\langle y_m(x), y_n(x) \rangle = \int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \underline{\underline{0}} \quad , m \neq n$$

* For SL problems, the eigenfunctions should be written in a form of complete set of functions (y_n) which yields to a set of basis functions defined on $0 \le x \le L$

* Generally, for a function $f(x)$, piecewise continuous function, and defined in $0 \le x \le L$, it can be written in terms of an infinite sum of basis functions " Fourier series "

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$$

" a_n is replacement of B "

where $$a_n = \frac{\langle f(x), \sin\frac{n\pi}{L}x \rangle}{\langle \sin\frac{n\pi}{L}, \sin\frac{n\pi}{L} \rangle}$$

$$\Rightarrow a_n = \frac{\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx}{\int_0^L \sin(\frac{n\pi}{L} . x) \sin(\frac{n\pi}{L} x) \, dx}$$

Nominator $\Rightarrow \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$ "$f(x)$ can be any function"

Denominator $\Rightarrow \int_0^L \sin \frac{n\pi}{L} x \sin \frac{n\pi}{L} x \, dx = \frac{L}{2}$

$$\Rightarrow a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Example $y'' + \lambda y = 0 \qquad 0 \le x \le L \, , \; L = 1$

BC's $y(0) - 2y'(0) = 0$

$y(L) = 0$

Sol'n $\Rightarrow$ Characteristic eq'n $r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda$

we will have 3 cases of λ

① $\lambda = 0 \Rightarrow$ The ODE becomes $y'' = 0$ "Integrate twice to find $y(x)$"

$y(x) = C + Dx$!

Let's apply BC's $\qquad y'(x) = D$

① $y(0) - 2y'(0) = 0$

$\Rightarrow \boxed{C - 2D = 0}$ —(1)

② $y(L=1) = 0 \Rightarrow \boxed{C + D = 0}$ —(2)

By Solving (1) and (2) $\Rightarrow$ $C = 0$ and $D = 0$

$\Rightarrow y(x) = 0$ "Trivial Sol'n" NOT a solution!

② $\lambda < 0 \Rightarrow r_{1,2} = \pm \sqrt{\lambda}$ "$r^2 = --\lambda \Rightarrow r^2 = \lambda$" "$r_{1,2} = \pm\sqrt{\lambda}$"

we will have two real distinct roots

Sol'n $\Rightarrow y(x) = A e^{\sqrt{\lambda} x} + B e^{-\sqrt{\lambda} x}$ $\Leftarrow$ this will never satisfy BC's "you can check"

③ $\lambda > 0 \Rightarrow r^2 = -\lambda \Rightarrow r_{1,2} = \pm i\sqrt{\lambda}$ "Two complex roots"

$y(x) = A \cos\sqrt{\lambda} x + B \sin\sqrt{\lambda} x$

Apply BC's

① $x = 0 \Rightarrow y(0) = 0 = A - 2(B\sqrt{\lambda}) \Rightarrow \boxed{A - 2B\sqrt{\lambda} = 0}$ —(1)

② $x = L = 1 \Rightarrow y(1) = 0 = A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda} \Rightarrow \boxed{A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda} = 0}$ —(2)

write in matrix Form

$$\begin{bmatrix} 1 & -2\sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$, this is an eigenvalue Problem! EVP

either $\begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ " trivial sol'n " (not a sol'n)

or $\det(m) = 0$, $[m] = \begin{bmatrix} 1 & -2\sqrt{\lambda} \\ \cos\sqrt{\lambda} & \sin\sqrt{\lambda} \end{bmatrix}$

$\Rightarrow \sin\sqrt{\lambda} + 2\sqrt{\lambda}\cos\sqrt{\lambda} = 0 \Rightarrow \tan\sqrt{\lambda} = -2\sqrt{\lambda}$ " we can obtain λ_n the eigenvalue"

therefore,

$\phi_n(x) = y_n(x) = 2\sqrt{\lambda_n}\cos\sqrt{\lambda_n}x + \sin\sqrt{\lambda_n}$

Eigenfunctions!

for $[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

" See the matrix operations review notes! "

Example Solve $y'' + \lambda y = 0$ $y(0) = 0$, $y(10) = 0$ $0 \le x \le 10$

Sol'n Characteristic eq'n $r^2 + \lambda = 0$ $r_{1,2} = \pm i\sqrt{\lambda}$ $\lambda > 0$

" we already know that for $\lambda \le 0$ there is no solution "

$\lambda > 0$ $r_{1,2} = \pm i\sqrt{\lambda}$

$\Rightarrow$ Sol'n becomes $y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

To find A and B, we use BC's

$y(0) = 0 = A \Rightarrow A = 0 \Rightarrow y(x) = B\sin\sqrt{\lambda}\,x$

$y(10) = 0 \Rightarrow B\sin\sqrt{\lambda}\,10 = 0 \Rightarrow \lambda_n = \frac{n\pi}{10}$ ← eignvalues

$y_n(x) = \sin\left(\frac{n\pi}{10}x\right)$ ← we know it's orthogonal

Therefore, $y(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{10}x$

$$a_n = \frac{\langle f(x), \sin\frac{n\pi}{10}x \rangle}{\langle \sin\frac{n\pi}{10}x, \sin\frac{n\pi}{10}x \rangle}$$

Practice Problem set 11.5 – page 503 (8, 9, 10, 11, 12, 13).