5.4 & 5.5 Bessel's Equation & Bessel's Functions

* **Bessel's Equation** $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$ → Eq(1) , ν: constant, real & $\nu \geq 0$

Compare to $x^2 y'' + x\, b(x) y' + c(x) y = 0 \Rightarrow b(x) = 1$ and $c(x) = x^2 - \nu^2$

Can we use Forbenious method? $b(0) = 1 \leftarrow b_0$ and $c(0) = -\nu^2 \leftarrow c_0$, both are analytic

So, yes, we can use F.M. Thus $y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$, derive y' and y''

then Subst. in ODE of Eq(1), to end up with the indicial equation:

$$r(r-1) + b_0 r + c_0 = 0 \quad \text{where } b_0 = 1 \text{ and } c_0 = -\nu^2$$

Thus: $r(r-1) + r - \nu^2 = 0 \Rightarrow r^2 = \nu^2$, remember ν is real ≥ 0

therefore, roots $r_1, r_2 = \pm \nu$

Root cases

- Case ①: For ν = integer = n $\quad r_1 = n$, $r_2 = -n$ "case 3 of F.M"

$$y_1(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! (n+m)!} = J_n(x)$$

$$y_2(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-n} m! (m-n)!} = J_{-n}(x)$$

General Sol'n : $y(x) = A \underbrace{J_n(x) + B J_{-n}(x)}$, A & B ⇒ constants (IC's)

↳ Bessel function of the 1st kind of order (n).

where $J_{-n}(x) = (-1)^n J_{(n)}(x)$, $n = 1, 2, \ldots$ " Int."

- Case ②: For $\nu \neq$ integer "Case 1 of F.M"

$$y_1(x) = x^{\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(m+\nu+1)} = J_{\nu}(x)$$

$$y_2(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(m-\nu+1)} = J_{-\nu}(x)$$

General Sol'n $y(x) = A \underbrace{J_{\nu}(x) + B J_{-\nu}}(x)$, A & B ⇒ IC's

↳ Bessel function of First kind of the order (ν)

* Gamma function $\Gamma(x)$ ⇒ A function similar to the factorial (!) but for non-integer values

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \, dt$$

* Special cases $\Gamma(n+1) = n!$, "n: Integer"
  - $\Gamma(3) = 2!$, $\Gamma(77) = 76!$
  - $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

* Important Properties of $J_\nu(x)$

(a) $$\left[x^{\nu} J_\nu(x)\right]' = x^{\nu} J_{\nu-1}(x)$$

(b) $$\left[x^{-\nu} J_\nu(x)\right]' = -x^{-\nu} J_{\nu+1}(x)$$

(c) $$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

(d) $$J_{\nu-1}(x) - J_{\nu+1}(x) = 2 J_\nu'(x)$$

(e) $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

(f) $$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

(g) $$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)$$

(h) $$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\cos x}{x} + \sin x\right)$$

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case 3 : $\nu = 0$ " repeated roots "

The ODE of Eq(1) becomes: $x^2 y'' + x y' + x^2 y = 0 \quad \div x \neq 0$

$$x y'' + y' + x y = 0$$

$$y(x) = y_1 + y_2 \quad \# \begin{cases} y_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}} \frac{x^{2m}}{m!} = J_0(x) \\ y_2(x) = J_0(x) \ln(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2^{2m}} \frac{h_m}{m!} x^{2m} = Y_0(x) \end{cases}$$

where $h_m = 1$ for $m = 1$

$h_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$, for $m = 2, 3, \dots$

$Y_0(x)$: Bessel function of the second type of the order zero.

* Relationship between J and Y " 1st and 2nd kinds Bessel functions"

For $\nu \neq$ Integer

$$Y_\nu = \frac{1}{\sin(\nu\pi)} \left(J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x) \right), \quad Y_{-\nu} \Rightarrow \text{replace } \nu \text{ by } (-\nu)$$

For ν = Integ. $= n$

$$Y_n = \lim_{\nu \to n} Y_\nu(x), \quad Y_{-n}(x) = (-1)^n Y_n(x)$$

Finally, the General solution of Bessel's Equation of ALL values of (ν) and $(x > 0)$

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

* Remember, we need Computers to obtain solutions of Bessel Eqn and Bessel's functions!

Practice
- Problem set 5.4 - page 195 $\Rightarrow$ (2, 3, 9 and 10)
- Problem set 5.5 - page 200 $\Rightarrow$ (1, 2, 7 and 9)

Example For the ODE: $x^2 y'' + x y' + (\lambda^2 x^2 - \nu^2) y(x) = 0$, λ: constant

① Show the this ODE can be reduced to Bessel's equation ($z = \lambda x$)

② Solve the ODE

Sol'n: $z = \lambda x \Rightarrow x = \frac{z}{\lambda}$, $x^2 = \frac{z^2}{\lambda}$, $y(z) = y(\lambda x)$

To find $\frac{dy}{dz}$, we use chain rule

$$y' = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \underbrace{\overset{\dot{y}}{\frac{dy}{dz}}} \cdot \lambda \quad \left(z = \lambda x,\ \frac{dz}{dx} = \lambda \right)$$

$$\Rightarrow \frac{dy}{dz} = \dot{y} = \frac{y'}{\lambda}$$

$$y'' = \frac{d^2 y}{dx} = \frac{d}{dx}\left(\frac{dy}{dz} \cdot \lambda\right) = \left(\frac{d}{dz}(\lambda \dot{y})\right) \frac{dz}{dx} \quad \text{"chain rule"}$$

$$= \lambda \underset{dy^2/dz^2}{\ddot{y}} \cdot \lambda \Rightarrow \boxed{y'' = \lambda^2 \ddot{y}}$$

Subst. in ODE

$$\underbrace{\frac{z^2}{\lambda^2}}_{x^2} \cdot \underbrace{\lambda^2 \ddot{y}}_{y''} + \underbrace{\frac{z}{\lambda}}_{x} \cdot \underbrace{\lambda \dot{y}}_{y'} + (\underbrace{z^2}_{x^2\lambda^2} - \nu^2)\, \underbrace{y(z)}_{y(\lambda x)} = 0$$

$$\Rightarrow \boxed{z^2 \ddot{y} + z \dot{y} + (z^2 - \nu^2) y(\lambda x) = 0} \longrightarrow \text{Bessel's Eq'n}$$

$$\Rightarrow y(z) = c_1 J_\nu(z) + c_2 Y_\nu(z)$$

or

$$y(\lambda x) = c_1 J_\nu(\lambda x) + c_2 Y_\nu(\lambda x)$$

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Chain rule

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \Rightarrow y' = \dot{y} \cdot \lambda$$

$$\frac{d^2 y}{dx^2} = \frac{d^2 y}{d^2 z} \cdot \left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz} \cdot \frac{d^2 z}{d^2 x} = \ddot{y} \lambda^2 + (\dot{y})(0) \Rightarrow \boxed{y'' = \ddot{y} \lambda^2}$$

* Modified Bessel's Equation – Section 5.5 – Problems 12 → 15

Modified Bessel's Equation $x^2 y'' + x y' + (-x^2 - \nu^2) y = 0$

To reduce to Bessel's equation, we need to do change of variables ($t = \hat{i} x$ or $x = -it$) and $y(x) = y(-\hat{i} t)$, $\hat{i} = \sqrt{-1}$

$t^2 = -x^2 \qquad x^2 = t^2$

therefore, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \hat{i}$ or $y' = \dot{y}\hat{i}$ $\left(\frac{dt}{dx} = \hat{i}\right)$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} \cdot \left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt} \cdot \frac{d^2t}{dx^2} \qquad \left(\frac{dt^2}{dx^2} = 0\right)$$

$$= \ddot{y}\,(-1)^2 + \dot{y}\,(0)$$

$$\boxed{y'' = \ddot{y}}$$

Subst. in ODE

$$t^2 \ddot{y} + -\hat{i} t (\hat{i} \dot{y}) + (t^2 - \nu^2) y(t) = 0$$

$$\Rightarrow \boxed{t^2 \ddot{y} + t \dot{y} + (t^2 - \nu^2) y = 0} \rightarrow \text{Bessel's eq.}$$

$y(t) = y(\hat{i} x)$

$$y(t) = A\, J_\nu(t) + B\, J_{-\nu}(t)$$

or

$$y(x) = A_0\, J_\nu(\hat{i} x) + B_0\, J_{-\nu}(\hat{i} x)$$

or

$$y(x) = C_1\, I_\nu(x) + C_2\, K_\nu(x)$$

where

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{x^{2m+\nu}}{2^{2m+\nu}\, m!\, \Gamma(m+\nu+1)}$$

Modified Bessel function of 1st Kind of order ν

and

$$K_\nu(x) = \sum_{m=0}^{\infty} \frac{x^{2m-\nu}}{2^{2m-\nu}\, m!\, \Gamma(m+\nu-1)}$$

Mod. Bessel function of 2nd Kind of order (ν)

$$= \frac{\pi}{2 \sin(\nu\pi)} \left(I_{-\nu}(x) - I_\nu(x) \right)$$