5.3 Forbenious Method

- For the 2nd order linear equation:

$$y'' + \frac{b(x)}{x} y' + \frac{C(x)}{x^2} y = 0 \qquad —(1)$$

- If $b(x)$ and $C(x)$ are analytic functions at $x=0$, the this ODE has at least one solution and expressed, as:-

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots)$$

where (r) can be a real or complex number. Also, (r) is selected so that $(a_0 \neq 0)$.

- Forbenious Method is commonly used to solve Bessel's equation as will be shown in next section.

- To use F.M to solve ODE's, we first multiply Eq(1) by x^2, thus:

$$x^2 y'' + x\, b(x)\, y' + C(x)\, y = 0 \qquad —(2)$$

to test the regularity of x_0, we need to test $x\,b(x)$ and $c(x)$ for $x = x_0$. If both functions are analytic "or has a value" then we can say that x_0 is a regular point of the ODE. If $x\,b(x)$ and/or $C(x)$ are non-analytic, then x_0 is a singular point of the ODE. For regular x_0, let

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y'(x) = \sum_{m=1}^{\infty} (m+r)\, a_m\, x^{(m+r-1)}$$

$$y''(x) = \sum_{m=2}^{\infty} (m+r)(m+r-1)\, a_m\, x^{m+r-2}$$

Substitute in ODE Eq(2)

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)\, a_m x^{m+r} + b(x) \sum_{m=0}^{\infty} (m+r)\, a_m x^{m+r} + C(x) \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

and let $\Rightarrow m = 0$

$$\Rightarrow r(r-1) + b(x)\, r + C(x) = 0$$

Expand $b(x)$ and $c(x)$ using Power series as:

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots \quad , \quad c(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

So $\Rightarrow$ $r(r-1) + r(b_0 + b_1 x + b_2 x^2 + \cdots) + (c_0 + c_1 x + c_2 x^2 + \cdots) = 0$

For $x = x_0 = 0$ $\Rightarrow$ $r(r-1) + b_0 r + c_0 = 0$ $\rightarrow$ Indicial equation

$\hookrightarrow$ 2nd order equation which has two roots.

Roots cases : Theorem 2 page 182-183 text book.

① Case 1 : Distinct roots not differing by an integer (r_1, r_2)

$$y(x) = y_1 + y_2 \Leftarrow \begin{cases} y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m \\ y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) = x^{r_2} \sum_{m=0}^{\infty} A_m x^m \end{cases}$$

② Case 2 : Double (repeated) roots $r_1 = r_2 = r = \frac{1}{2}(1 - b_0)$

$$y(x) = y_1 + y_2 \Leftarrow \begin{cases} y_1(x) = x^{r}(a_0 + a_1 x + a_2 x^2 + \cdots) = x^{r} \sum_{m=0}^{\infty} a_m x^m \\ y_2(x) = y_1(x)\ln(x) + x^{r}(A_0 + A_1 x + A_2 x^2 + \cdots) = y_1(x)\ln(x) + x^{r} \sum_{m=0}^{\infty} A_m x^m \end{cases}$$

③ Case 3 : Distinct roots seperated by an integer (r_1, r_2 where $r_1 - r_2 = \text{int.}$) $r_1 > r_2$

$$y(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m$$

$$y_2(x) = K y_1(x)\ln(x) + x^{r_2}(A_0 + A_1 x + A_2 x + \cdots) = K y_1(x)\ln(x) + x^{r_2} \sum_{m=0}^{\infty} A_m x^m$$

K : constant and maybe equal to zero.

Example Solve $6x^2 y'' + 7xy - (1 + x^2) y = 0$

sol. Divide by 6 $\Rightarrow$ $x^2 y'' + \underbrace{\frac{7}{6} x}_{x\, b(x)} y - \underbrace{\frac{1+x^2}{6}}_{c(x)} y = 0$

$b(x) = \frac{7}{6}$

$c(x) = \frac{-(1+x^2)}{6}$ $\Rightarrow$ $b(0) = \underbrace{\frac{7}{6}}_{b_0}$, $c(0) = \underbrace{\frac{-1}{6}}_{c_0}$ "Analytic at $x=0$"

So, using the indicial equation

$$r(r-1) + b_0 r + c_0 = 0 \Rightarrow r(r-1) + \frac{7}{6} r - \frac{1}{6} = 0$$

$\Rightarrow$ $r_1 = \frac{1}{3}$, $r_2 = \frac{-1}{2}$ " Distinct roots with no separation by integer "

Therefore, $y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m = x^{1/3} \sum_{m=0}^{\infty} a_m x^m$

$$y_2(x) = x^{r_2} \sum_{m=0}^{\infty} A_m x^m = x^{-1/2} \sum_{m=0}^{\infty} A_m x^m$$

To obtain a_m & A_m, substitute $y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$ in the ODE:

$$6x^2 \sum_{m=2}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2} + 7x \sum_{m=1}^{\infty} a_m (m+r) x^{m+r-1} - (1+x^2) \sum_{m=0}^{\infty} a_m x^m = 0$$

Expand and Distribute:

$$6 \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + 7 \sum_{m=0}^{\infty} a_m (m+r) x^{m+r} - \sum_{m=0}^{\infty} a_m x^{m+r} - \sum_{m=-2}^{\infty} a_m x^{m+r+2} = 0$$

$\sum_{m=-2}^{\infty} a_m x^{m+r+2}$, let $n = m+2 \Rightarrow \sum_{n=0}^{\infty} a_{n-2} x^{n+r}$, return $n=m \Rightarrow \sum_{m=0}^{\infty} a_{m-2} x^{m+r}$

$$\Rightarrow 6 \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) \cancel{x^{m+r}} + 7 \sum_{m=0}^{\infty} a_m (m+r) \cancel{x^{m+r}} - \sum_{m=0}^{\infty} a_m \cancel{x^{m+r}} - \sum_{m=0}^{\infty} a_{m-2} \cancel{x^{m+r}} = 0$$

Collect a_m

$$\sum_{m=0}^{\infty} \Big(6(m+r)(m+r-1) + 7(m+r) - 1\Big) a_m - \sum_{m=0}^{\infty} a_{m-2} = 0$$

The terms inside Σ should be equal to zero, to satisfy the equation

thus, $\Big(6(m+r)(m+r-1) + 7(m+r) - 1\Big) a_m - a_{m-2} = 0$

$$a_m = \frac{a_{m-2}}{6(m+r)(m+r-1) + 7(m+r) - 1}$$

For $r = \frac{1}{3} \Rightarrow a_m = \dfrac{a_{m-2}}{(m+\frac{1}{3})\left[6(m-\frac{2}{3})+7\right]-1}$ — (*)

$r = -\frac{1}{2} \Rightarrow a_m = \dfrac{a_{m-2}}{(m-\frac{1}{2})\left[6(m-\frac{3}{2})+7\right]-1}$, or "$A_m$" — (**)

$y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m = x^{1/3} \sum_{m=0}^{\infty} a_m x^m$, a_m is of Eq(*)

$y_2(x) = x^{r_2} \sum_{m=0}^{\infty} A_m x^m = x^{-1/2} \sum_{m=0}^{\infty} A_m x^m$, A_m is of Eq(**)

$$y(x) = y_1 + y_2$$ #

Example $x(x-1)y'' + (3x-1)y' + y = 0$ original

1. Proof that we can use Forbenious method to solve ODE
2. If so, what are $b(x)$ and $c(x)$?
3. Solve the ODE.

Sol. Divide by $x(x-1) \Rightarrow y'' + \dfrac{3x-1}{x(x-1)} y' + \dfrac{1}{x(x-1)} y = 0$

Multipy by $x^2 \Rightarrow x^2 y'' + x \dfrac{3x-1}{x-1} y' + \dfrac{x}{x-1} y = 0$

compare to $x^2 y'' + x\, b(x) y' + c(x) y = 0$

$$\Rightarrow b(x) = \frac{3x-1}{x-1} \qquad c(x) = \frac{x}{x-1}$$

Analytic at $x=0$? $b(0) = 1$, $c(0) = 0$ — Yes!

we can use F.M and $b(x) = \dfrac{3x-1}{x-1}$, $c(x) = \dfrac{x}{x-1}$!

③ Solution of ODE $\Rightarrow$ Inicial equation $r(r-1) + \overset{1}{b_0} r + \overset{0}{c_0} = 0$

So $\Rightarrow r(r-1) + r = 0 \Rightarrow$ roots $r_1, r_2 = r = 0$ "double root"

For double roots $y_1(x) = x^r \sum_{m=0}^{\infty} a_m x^m$

$$y_2(x) = y_1(x)\ln(x) + x^r \sum_{m=0}^{\infty} A_m x^m$$

To find a_m and A_m, subst. $y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$ in ODE (original)

$$\Rightarrow \sum_{m=0}^{\infty} (m+r)(m+r-1) x^{m+r} - \sum_{m=1}^{\infty} (m+r)(m+r-1) a_m x^{m+r-1} + 3\sum_{m=0}^{\infty} (m+r) a_m x^{m+r}$$
$$- \sum_{m=1}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

let $n = m-1$, $m=1, n=0$

$$\to \sum_{n=0}^{\infty} (n+r+1)(n+r) a_{n+1} x^{n+r} \Rightarrow \sum_{m=0}^{\infty} (m+r)(m+r+1) a_{m+1} x^{m+r}$$

$$\to \sum_{n=0}^{\infty} (n+r+1) a_{n+1} x^{n+r} \Rightarrow \sum_{m=0}^{\infty} (m+r+1) a_{m+1} x^{m+r}$$

Therefore

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m \cancel{x^{m+r}} - \sum_{m=0}^{\infty} (m+r)(m+r+1) a_{m+1} \cancel{x^{m+r}} + 3\sum_{m=0}^{\infty} (m+r) a_m \cancel{x^{m+r}}$$
$$- \sum_{m=0}^{\infty} (m+r+1) a_{m+1} \cancel{x^{m+r}} + \sum_{m=0}^{\infty} a_m \cancel{x^{m+r}} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left\{ \left[(m+r)(m+r-1) + 3(m+r) + 1\right] a_m + \left[-(m+r)(m+r+1) - (m+r+1)\right] a_{m-1} \right\} = 0$$

To satisfy this eq, the terms inside Σ must $=0$

$$\left[(m+r)(m+r-1)+3m+1\right]a_m+\left[-(m+r)(m+r+1)-(m+r+1)\right]a_{m+1}=0$$

Remember $r=0$, Subst. above, thus $\boxed{a_{m+1}=a_m}$, $a_0=a_1=a_2=\cdots$

Let $a_0=1 \Rightarrow a_m=1$

therefore: $y_1(x)=x^r\sum_{m=0}^{\infty}a_m x^m \xRightarrow{r=0,\ a_m=1} y_1(x)=\sum_{m=0}^{\infty}x^m$

$$y_2(x)=y_1(x)\ln(x)+x^r\sum_{m=0}^{\infty}A_m x^m$$, here $a_m=A_m=1$, $r=0$

$$y_2(x)=y_1(x)\ln(x)+x^0\underbrace{\sum_{m=0}^{\infty}a_m x^m}_{y_1(x)}$$

$$y_2(x)=y_1(x)\left(\ln(x)+1\right)$$

$$\boxed{y=y_1+y_2}$$

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