Chapter 3 Higher order ODE's

3.1 Homogeneous linear ODE's

An n^{th} order non-Homog. ODE (linear) can be generally expressed as:

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \dots + P_1(x)\, y' + P_0(x)\, y = r(x) \quad , \quad y^{(n)} = \frac{dy^n}{dx^n}$$

where $P_0, P_1, \dots, P_{n-1}, P_n$ and $r(x)$ are any given functions of (x).
The derivative $y^{(n)}$ has a coefficient of 1. This is the standard form.

Note 1: For $n=2$, the ODE becomes 2nd order where $P_1(x) = p(x)$ and $P_0(x) = q(x) \Rightarrow y'' + p(x)y' + q(x)\, y = r(x)$.

Note 2: For $r(x) = 0$, the ODE becomes homogn. and written as:-

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \dots + P_1(x)\, y' + P_0(x)\, y = 0$$

3.2 Homogeneous ODE's with constant coefficients

The n^{th} order linear homog. ODE has constant coefficients is:

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \quad , \quad y^{(n)} = \frac{dy^n}{dx^n}$$

To solve this ODE, let $y(x) = e^{\lambda x}$ and substitute in the ODE, so:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0 \qquad \text{"characteristic eq'n"}$$

Based on the roots, we will have four cases

Case 1: Real Distinct roots

If the roots of the Ch. eq are all real and distinct $(\lambda_1, \lambda_2, \dots \lambda_n)$, the solutions are

$$y_1(x) = e^{\lambda_1 x}, \; y_2(x) = e^{\lambda_2 x}, \; \dots, \; y_n(x) = e^{\lambda_n x}$$

Using the superposition principle, the general solution becomes:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_{n-1} y_{n-1}(x) + c_n y_n(x)$$

where $c_1, c_2, \dots, c_{n-1}, c_n$ are constants to be determined from the initial conditions.

Example Solve $y''' + 2y'' - y' + 2y = 0$

Sol. charact. eq $\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

roots $\lambda_1 = -1, \; \lambda_2 = 1, \; \lambda_3 = 2$

Thus, $y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$

c_1, c_2 and $c_3 \Rightarrow$ From IC's.

Case 2 Simple complex roots

For complex roots, and written in conjugate pairs $(\lambda_n = \alpha \pm \hat{i}\beta)$, $\hat{i} = \sqrt{-1}$

the solutions would be:

$y_1(x) = A e^{\alpha x}(\cos \beta x)$, $y_2(x) = B e^{\alpha x}(\sin x)$, ...

the general sol. (superposition) becomes

$y(x) = e^{\alpha x}(A \cos \beta x + B \sin \beta x) + \cdots$

Example: $y''' - y'' + 100y' - 100y = 0$, $y(0) = 4$, $y'(0) = 11$ and $y''(0) = 299$

Sol. ch. Eq'n $\Rightarrow \lambda^3 - \lambda^2 + 100\lambda - 100 = 0$

roots $\Rightarrow \lambda_1 = 1$, $\lambda_2 = +10\hat{i}$, $\lambda_3 = -10\hat{i}$ $\quad \alpha = 0$, $\beta = 10$

λ_1 ↳ real: $y_1(x) = C_1 e^x$

$\lambda_{2,3}$ ↳ complex roots: $y_{2,3} = A \cos 10x + B \sin x$

Total Sol. $y(x) = C_1 e^x + A \cos 10x + B \sin x$

To find C_1, A and B $\Rightarrow$ we use IC's

therefore $\Rightarrow C_1 = 1$, $A = 3$, $B = 1$

$$y(x) = e^x + 3\cos 10x + \sin 10x$$

Case 3: Multiple (repeated) real roots

If we have real double roots $\lambda_1 = \lambda_2 = \lambda$, then:

$y_1(x) = e^{\lambda x}$, $y_2 = x e^{\lambda x}$

Generally, for "m" double roots (λ)

$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x} + C_3 x e^{\lambda x} + \cdots + C_n x^{m-1} e^{\lambda x}$

Example: Solve $y^{v} - 3y^{iv} + 3y''' - y'' = 0$

Sol. char. Eq'n $\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$

Roots $\lambda_1 = 0$, $\lambda_2 = 0$ $\Rightarrow y_1(x) = C_1 e^{0x} = C_1$, $y_2(x) = C_2 x e^{0x} = C_2 x$

$\lambda_3 = \lambda_4 = \lambda_5 = 1 \Rightarrow y_3(x) = C_3 e^x$, $y_4(x) = C_4 x e^x$, $y_5(x) = C_5 x^2 e^x$

$\Rightarrow y(x) = C_1 + C_2 x + e^x(C_3 + C_4 x + C_5 x^2)$ #

Case 4 Multiple (repeated) complex roots

If $\lambda_1, \lambda_2 = \alpha_0 \pm \hat{i}\beta_0 \Rightarrow y_{1,2} = e^{\alpha_0 x}(A_0 \cos \beta_0 x + B_0 \sin \beta_0 x)$

and $\lambda_{3,4} = \alpha_0 \pm \hat{i}\beta_0 \Rightarrow y_{3,4} = \underline{x}\, e^{\alpha_0 x}(A_1 \cos \beta_0 x + B_1 \sin \beta_0 x)$

$$y(x) = e^{\alpha_0 x}\left[A_0 \cos \beta_0 x + B_0 \sin \beta_0 x\right] + x e^{\alpha_0 x}\left[A_1 \cos \beta_0 x + B_1 \sin \beta_0 x\right]$$

or,

$$y(x) = e^{\alpha_0 x}\left[(A_0 + A_1 x)\cos \beta_0 x + (B_0 + B_1 x)\sin \beta_0 x\right]$$

Practice: Problem Set 3.2 page 116 (1→3, 7, 9, 11, 13) #

3.3 Non-Homogeneous ODE'S

For nth order ODES and non-homog. with variable coefficients:

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \cdots + P_1(x) y' + P_0(x) y = r(x) \quad , \quad r(x) \neq 0$$

Total Sol. $y(x) = \underbrace{y_h(x)}_{\text{homog. Sol'n}} + \underbrace{y_p(x)}_{\text{particular (non-homog.) sol'n}}$

To find the $y_p(x)$:

① Method of undetermined coefficents
② method of variation of Parameters

} as ch.2

① Undetermined Coefficients

As in Ch.2, the solution of $y_p(x)$ can be obtained as:

1- Find homog. Sol'n $y_h(x)$

2- Assume $y_p(x)$, see the table in ch.2

3- For any repeated Solution, we multipy by (x)

4- Find the coefficients of $y_p(x)$.

Example Solve $y^{iv} - y = 4.5\, e^{-2x}$

Sol. ① Homog. Sol'n $y_h(x)$, $y'''' - y = 0$

Charact. eq'n $\Rightarrow$ $\lambda^4 - 1 = 0$, roots $\Rightarrow$ $\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = +\hat{i},\ \lambda_4 = -\hat{i}$

$$\Rightarrow y_h(x) = C_1 e^{x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x$$

② Non-Homog. Sol'n $y_p(x) = C\, e^{-2x}$ ← similar to $r(x)$

Substitute in ODE $y'''' - y = 4.5 e^{-2x}$

$$(-2)^4\, C \cancel{e^{-2x}} - C \cancel{e^{-2x}} = 4.5\, \cancel{e^{-2x}}$$

$$\Rightarrow C = 0.3 \quad \Rightarrow \quad y_p(x) = 0.3\, e^{-2x}$$

Total Sol. $y(x) = y_h + y_p$

$$y(x) = C_1 e^{x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x + 0.3\, e^{-2x}$$

If we have IC's $\Rightarrow$ obtain C_1, C_2 and C_3

#

Example Solve $y''' - 3y'' + 3y' - y = 30e^x$

Sol. ① Homog. Sol. $\Rightarrow y''' - 3y'' + 3y' - y = 0$

char. Eq'n $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

roots $\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1$ ← Triple repeated roots

$$y_h(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

② Non-Homog. Sol'n $y_p(x) = C\, x^3\, e^x$

↳ because we need independ. Sol'n

Subst. in ODE

$$y''' - 3y'' + 3y' - y = 30e^x$$

we will get $y_p(x) = 5x^3 e^x$ "C=5"

$y_p(x$

Total Sol'n $y(x) = y_h + y_p$

$$y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + 5x^3 e^x$$

c_1, c_2 and $c_3 \Rightarrow$ from IC's.

② Variation of Parameters

For a linear non-homog. nth order ODE :-

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \cdots + P_1(x)\, y' + P_0(x)\, y = r(x), \qquad r(x) \neq 0$$

we use the method of Variation of Parameters to find the Particular Solution $(y_p(x))$, as:-

$$y_p(x) = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\, dx + y_2(x) \int \frac{W_2(x)}{W(x)} r(x)\, dx + \cdots + y_n \int \frac{W_n(x)}{W(x)} r(x)\, dx$$

Where $y_1(x), y_2(x), \ldots, y_n(x)$ are the Solutions of the homogeneous ODE:

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \cdots + P_1(x)\, y' + P_0\, y = 0 \leftarrow \text{homog.}$$

and $W(x)$ is the Wronskain and $W_i(x)$ $\{i = 1, 2, \ldots, n\}$ is obtained by replacing ith column of $W(x)$ by $[0\ 0\ \cdots\ 1]^T$. For example, $n=2$ "2nd order ODE"

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1(x) = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} \quad \text{and} \quad W_2(x) = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} \quad \#$$

Example: Solve $y''' - 2y'' - y' + 2y = \ln(x)$

Sol. ① Homog. Sol'n $\Rightarrow y''' - 2y'' - y' + 2y = 0$

Charact Eq'n $\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 2$ $\left\{ \begin{array}{l} \text{real} \\ \& \\ \text{Distinct} \end{array} \right\}$

$$\Rightarrow y_h(x) = c_1 \underbrace{e^{-x}}_{y_1} + c_2 \underbrace{e^{x}}_{y_2} + c_3 \underbrace{e^{2x}}_{y_3} \qquad , \quad r(x) = \ln(x)$$

Non-Homog. Sol'n ② $y_p(x) = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\,dx + y_2 \int \frac{W_2(x)}{W(x)} r(x)\,dx + y_3 \int \frac{W_3(x)}{W(x)} r(x)\,dx$

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \xleftarrow{\text{determinate}} = \begin{vmatrix} e^{-x} & e^{x} & e^{2x} \\ -e^{-x} & e^{x} & 2e^{2x} \\ e^{-x} & e^{x} & 4e^{+2x} \end{vmatrix}$$

$$W(x) = 2e^{x}\left(e^{2x} + 2\right)$$

$$W_1(x) = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ 1 & e^{x} & 4e^{2x} \end{vmatrix} , \quad W_2(x) = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-x} & 0 & e^{2x} \\ -e^{-x} & 0 & 2e^{2x} \\ e^{-x} & 1 & 4e^{2x} \end{vmatrix}$$

$$W_3(x) = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{x} & 0 \\ -e^{-x} & e^{x} & 0 \\ e^{-x} & e^{x} & 1 \end{vmatrix} .$$

The final $y_p \Rightarrow$

$$y_p(x) = e^{-x} \int \frac{W_1(x)}{W(x)} \ln(x)\,dx + e^{x} \int \frac{W_1(x)}{W(x)} \ln(x)\,dx + e^{2x} \int \frac{W_3(x)}{W(x)} \ln(x)\,dx$$

The total solution

$$y(x) = y_h + y_p$$

* For Another example, see example 2 – Page 119 " text book "

* Practice: Problem set 3.3 – Page 122 (1, 2, 8, 10)

End of chapter 3