Section 2.7 Non-Homogeneous ODE's

A second order linear non-Homogeneous ODE is:

$$y'' + p(x)y' + q(x)y = r(x) \quad , \quad r(x) \neq 0$$

The general solution for non-Homog ODE's

$$\underbrace{y(x)}_{\text{Total Sol.}} = \underbrace{y_h(x)}_{\text{homog. sol.}} + \underbrace{y_p(x)}_{\text{Particular solution or non-Hom. Sol.}} \quad , \quad y_p \text{ and } y_h \rightarrow \text{Linearly independent}$$

The homog. sol. can be obtained as we have shown before. Now, we focus on the particular solution ($y_p(x)$). This is usually achieved by assigning a suitable form of $y_p(x)$ that is similar to $r(x)$.

Example: Solve $\Rightarrow y'' - 4y' + 3y = 10e^{-2x}$ $\quad y(0) = 1$, $y'(0) = -3$

Sol. $\quad y(x) = y_h(x) + y_p(x)$

① Homog. sol $\Rightarrow y'' - 4y' + 3y = 0$

Character. eq $\Rightarrow r^2 - 4r + 3 = 0 \Rightarrow$ $\left.\begin{array}{l} r_1 = 1 \\ r_2 = 3 \end{array}\right\}$ Distinct roots

$$y_h(x) = c_1 e^{x} + c_2 e^{3x}$$

② Non-Homog. sol. $\Rightarrow y'' - 4y' + 3y = 10e^{-2x}$

Assume $y_p(x) = ce^{-2x} \leftarrow$ Similar to $r(x) = 10e^{-2x}$

and subst. in ODE $\quad [\, y'(x) = -2ce^{-2x} \;,\; y''(x) = 4ce^{-2x} \,]$

So, $\Rightarrow 4c\cancel{e^{-2x}} + 8c\cancel{e^{-2x}} + 3c\cancel{e^{-2x}} = 10\cancel{e^{-2x}}$

$\Rightarrow (4+8+3)c = 10 \quad \Rightarrow c = \frac{10}{15} = \frac{2}{3}$

So, $\quad y_p(x) = \frac{2}{3}e^{-2x}$

Total Sol. $\quad y(x) = y_h + y_p \Rightarrow y(x) = c_1 e^{x} + c_2 e^{3x} + \frac{2}{3}e^{-2x}$

To find c_1 and c_2 $\Rightarrow$ Initial conditions

$y(x) = C_1 e^{x} + C_2 e^{3x} + \frac{2}{3} e^{-2x}$, $y(0) = 1$, $y'(0) = -3$ ②

First IC $\Rightarrow$ $y(0) = 1 = C_1 + C_2 + \frac{2}{3} \Rightarrow C_1 + C_2 = 1/3$

2nd IC $\Rightarrow$ $y'(0) = -3 = C_1 + 3C_2 - 4/3 \Rightarrow C_1 + 3C_2 = -\frac{5}{3}$

$y'(x) = C_1 e^{x} + 3C_2 e^{3x} - \frac{4}{3} e^{-2x}$

Two eq'ns, two unkn
$\left.\begin{array}{l} C_1 = 4/3 \\ C_2 = -1 \end{array}\right.$

Final sol. $y(x) = \frac{4}{3} e^{x} - e^{3x} + \frac{2}{3} e^{-2x}$

* The method shown in this example is called " method of Undetermined coefficients" in which we assume the $y_p(x)$ in a form that is similar to $r(x)$ and generally used for ODE's with constant coefficients " $y'' + ay' + by = r(x)$".

Procedure $y(x) = y_h + y_p$

① Obtain homog. solution as we learned.

② Assume $y_p(x)$ " non-homog." in a combination between all linear independent functions of $r(x)$, generated by differentiation.

Important → ③ Any duplicate terms of $y_h(x)$ should be multiplied by x.

④ Substitute this y_p in the ODE to obtain constants as we did in the previous example.

⑤ combine y_p and y_h to find total sol. $y(x)$ then use IC's to obtain the resulting constants.

$y_p(x)$ forms and selections

$r(x)$	$y_p(x)$
Ke^{ax}	Ce^{ax}
Kx^n, $n = 0, 1, 2, \ldots$	$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_1 x + C_0$
$K \cos wx$ $K \sin wx$	$A \cos wx + B \sin wx$
$Ke^{ax} \cos wx$ $Ke^{ax} \sin wx$	$e^{ax}(A \cos wx + B \sin wx)$

← Very Important!

Example: Solve $y'' + 4y = 8x^2$

Sol. $y(x) = y_h + y_p$

$\alpha = 0$
$\beta = 2$

① $y_h \Rightarrow \quad r^2 + 4 = 0 \Rightarrow r_{1,2} = \pm \hat{i}\,2 \leftarrow$ complex

$y_h(x) = A_1 \cos 2x + B \sin 2x$

② $y_p(x) = a_2 x^2 + a_1 x + a_0 \leftarrow$ Subst. in ODE

$y_p'(x) = 2a_2 x + a_1$
$y_p''(x) = 2a_2$

$\Rightarrow \quad 2a_2 + 4(a_2 x^2 + a_1 x + a_0) = 8x^2$

$4a_2 x^2 + 4a_1 x + (4a_0 + 2a_2) = 8x^2$, Now, we collect terms.

$x^2 \Rightarrow 4a_2 \cancel{x^2} = 8\cancel{x^2} \Rightarrow a_2 = 2$

$x \Rightarrow 4a_1 x = 0 \Rightarrow a_1 = 0$

const $\Rightarrow 4a_0 + 2a_2 = 0 \Rightarrow a_0 = \frac{-2a_2}{4} \Rightarrow a_0 = -1$

so. $\boxed{y_p(x) = 2x^2 - 1}$

Total solution $y(x) = y_h + y_p$

$y(x) = A \cos 2x + B \sin 2x + 2x^2 - 1$

If we have IC's, we can obtain A & B.

Example: Solve $y'' - 3y' + 2y = e^x$

Sol. ① $y_h(x) \Rightarrow r^2 - 3r + 2 = 0 \Rightarrow \left. \begin{matrix} r_1 = 1 \\ r_2 = 2 \end{matrix} \right\} y_h(x) = C_1 e^x + C_2 e^{2x}$

② $y_p(x) \Rightarrow$ Assume $y_h(x) = C x e^x$

↳ Similar to $r(x) = e^x$ but multiplied by $\underline{\underline{x}}$ because same as $C_1 e^x$

↳ Subst. in ODE $\Rightarrow$

$C(2+x)\cancel{e^x} - 3C(1+x)\cancel{e^x} + 2Cx\cancel{e^x} = \cancel{e^x}$

$C = -1 \Rightarrow y_p(x) = -xe^x$

total sol. $y(x) = y_h + y_p \Rightarrow y(x) = C_1 e^x + C_2 e^{2x} - xe^x$

C_1 and C_2 from IC's

II) Method of Variation of Parameters

This method is commonly used for 2nd order ODE's with "Variable" coefficients $y'' + P(x)y' + q(x)y = r(x)$

- Solution Procedure

① Find homog. Sol. $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

② Find $y_p(x) = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$

where $W = y_1 y_2' - y_2 y_1'$ "Wronskain $\neq 0$"

Example: Solve $y'' + y = \sec(x)$

Solution

① $y_h \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm \hat{i}$ → complex roots $\alpha = 0$, $\beta = 1$

$$y_h(x) = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$$

② $y_p(x) = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$

$W = y_1 y_2' - y_2 y_1' = \cos^2 x + \sin^2 x = 1$

So, $y_p(x) = -\cos x \int \sin x \cdot \sec x \, dx + \sin x \int \cos x \cdot \sec x \, dx$

$y_p(x) = \cos x \ln|\cos x| + x \sin x$

Total Sol

$$y(x) = y_h + y_p$$
$$= c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

c_1 and c_2 constants ⇒ IC's

* Practice Problem Set 2.7 page 84-85 from textbook
1 → 5, 8, 13, 14