2.5 Cauchy–Euler Equation

We know that the general form of a linear 2nd order ODE is:

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0 \qquad \text{(Homogeneous)}$$

* Special case: Cauchy–Euler Eq.

$$a_0(x) = x^2, \quad a_1(x) = ax, \quad a_2(x) = b$$

a and b constants

Thus, $x^2 y'' + a x y' + b y = 0$, to solve: let $y(x) = x^m$, $x^m \neq 0$ non-trivial

So, substitute $y = x^m$, $y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$

$$\Rightarrow x^2\, m(m-1)x^{m-2} + a x\, m x^{m-1} + b x^m = 0$$

$$\Rightarrow m(m-1)x^m + a m x^m + b x^m = 0, \quad \text{divide by } x^m$$

$$\Rightarrow m^2 + (a-1)m + b = 0$$

Auxiliary equation "2nd order polynom"

→ we can find roots m_1 and m_2

$$ax^2 + bx + c = 0$$

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Root cases

① Two Real distinct roots (m_1 & m_2)

$$y(x) = C_1 x^{m_1} + C_2 x^{m_2}, \quad C_1 \,\&\, C_2 : \text{constants}$$

Don't forget the Principle of superpos.

② Repeated roots $m_1 = m_2 = m$, $m = \frac{1}{2}(1-a)$

$$y(x) = \left(C_1 + C_2 \ln(x)\right) x^m, \quad m = \tfrac{1}{2}(1-a)$$

③ Complex roots $m_{1,2} = \underset{\text{real}}{\alpha} \pm \hat{i}\, \underset{\text{Im}}{\beta}$, $\hat{i} = \sqrt{-1}$

$$y(x) = x^{\alpha}\left(A \cos(\beta \ln(x)) + B \sin(\beta \ln(x))\right)$$

Example: Solve $x^2 y'' + 7x y + 13 y = 0$

Sol. $a_0 = x^2$, $a_1 = \underset{ax}{7x}$, $a_2(x) = \underset{b}{13}$ $\Rightarrow$ "Cauchy-Euler"

To Solve, auxiliary eq.

$$m^2 + (a-1)m + b = 0 \quad , \quad \begin{matrix} a = 7 \\ b = 13 \end{matrix}$$

So, $m^2 + 6m + 13 = 0$

roots $m_{1,2} = \dfrac{-6 \pm \sqrt{36-(4)(13)}}{2}$, $\left.\begin{matrix} m_1 = -3+2i \\ m_2 = -3-2i \end{matrix}\right\}$ complex roots

$\alpha = -3$
$\beta = 2$

Solution $y(x) = x^{\overset{\text{real}}{-3}} \left[A \cos(2 \ln x) + B \sin(2 \ln x) \right]$

A, B, we need IC's.

* practice Problem Set 2.5 page 73 from text book
3, 5, 10, 14

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