Chapter 2 : Second order linear ODE's

2.1

A second order ODE is called <u>linear</u> if it can be written:

$\leftarrow$ 2nd order

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x) y = C(x) \quad , \text{ where } a_0(x) \neq 0$$

If we divide by $a_0(x)$, thus

$$y'' + p(x)\, y' + q(x) y = r(x)$$

$a_1/a_0 = p$, $\frac{C(x)}{a_0} = r(x)$

$a_2/a_0 = q$

Two cases – ① Homogenous ODE $r(x) = 0$

$$y'' + p(x) y' + q(x)\, y = 0$$

– ② Non-Homogeneous ODE , $r(x) \neq 0$

$$y'' + p(x) y' + q(x) y = r(x)$$

* Special case, homogenous ODE where $p(x)$ and $q(x)$ are constants:

$$y'' + a y' + by = 0 \quad , \text{ a and b constants.}$$

(Sec 2.2 text book)

To solve: let $y(x) = C e^{rx}$ C: constant $\neq 0$

and substitute in ODE. $y'(x) = r C e^{rx}$, $y'' = r^2 C e^{rx}$

So, $r^2 \cancel{C}\, \cancel{e^{rx}} + a r \cancel{C}\, \cancel{e^{rx}} + b \cancel{C}\, \cancel{e^{rx}} = 0$

$$\Rightarrow \boxed{r^2 + ar + b = 0} \rightarrow \text{characteristic equation}$$

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$ax^2 + bx + c = 0$

$x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow$ Two solutions $y_1(x) = C_1 e^{r_1 x}$, $y_2(x) = C_2 e^{r_2 x}$

Using the principle of superposition " Any linear combination of two solutions is also a solution "

$$y(x) = y_1(x) + y_2(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Solution Cases

① Two distinct real roots $(a^2 - 4b > 0)$

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$, c_1, c_2 constants

② Repeated roots $r_1 = r_2 = r$ $(a^2 - 4b = 0)$

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$ c_1 & c_2 constants

③ Complex roots $(a^2 - 4b < 0)$

$r_1, r_2 = \alpha \pm i\beta$ $\quad i = \sqrt{-1}$

$$y(x) = e^{\alpha x}(A \cos\beta x + B \cos\beta x)$$ A, B contants.

Example:- Solve $y'' + y' - 2y = 0$, IC's : $y(0) = 4$, $y'(0) = -5$

Sol $y(x) = ce^{rt}$ subst. , $y' = cre^{rt}$, $y'' = cr^2 e^{rt}$

$$\cancel{c} r^2 \cancel{e^{rt}} + \cancel{c} r \cancel{e^{rt}} - 2\cancel{c}\cancel{e^{rt}} = 0 \Rightarrow \boxed{r^2 + r - 2 = 0}$$

roots $r_1, r_2 = \dfrac{-1 \pm \sqrt{1 - (4)(2)}}{2} \Rightarrow r_1 = 1$, $r_2 = -2$ " Distinct roots "

So, $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \Rightarrow y(t) = c_1 e^{t} + c_2 e^{-2t}$

To find c_1 and c_2 ⇒ Initial conditions.

① $y(0) = 4 = c_1 e^{0} + c_2 e^{-2(0)} \Rightarrow \boxed{c_1 + c_2 = 4}$

② $y'(0) = -5 = c_1 e^{0} - 2c_2 e^{-2(0)}$, $y'(x) = c_1 e^{t} - 2c_2 e^{-2t}$

$\Rightarrow \boxed{c_1 - 2c_2 = -5}$

Two equations and two unknowns ⇒ $c_1 = 1$, $c_2 = 3$ ⇒ $y(x) = e^{t} + 3e^{-2t}$

Example: Solve $y'' - 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 1$

Sol

$r^2 - 4r + 4 = 0$ ⇒ roots $r_1 = r_2 = 2$ " repeated roots "

$y(t) = c_1 e^{rt} + c_2 t e^{rt}$

$y(t) = c_1 e^{2t} + c_2 t e^{2t}$, $c_1 = 3$, $c_2 = -5$ "From IC's"

$$y(t) = 3e^{2t} - 5e^{2t}$$

Example: Solve $y'' - 2y' + 10y(t) = 0$

Sol $\Rightarrow r^2 - 2r + 10 = 0$, $r_1 = 1 + 3\hat{i}$, $r_2 = 1 - 3\hat{i}$ "complex roots"

For complex roots

$$y(t) = e^{\alpha t}\left(A \cos \beta t + B \sin \beta t\right), \quad r_1, r_2 = \underbrace{\alpha}_{\text{real}} \pm i \underbrace{\beta}_{\text{Imag}}$$

So,

$$y(t) = e^{t}\left(A \cos 3t + B \sin 3t\right), \quad \alpha = 1, \ \beta = 3$$

If we have IC's, we can obtain A & B.

* Practice Problem set 2.2 from textbook page 59

2, 6, 11, 12, 13, 22 and 26, and 29