Chapter 1: First Order ODE's

1.1 & 1.2

The first (1st) order ODE is the simplest form which mainly contain only y' or $\frac{dy}{dx}$ and expressed as $g(x, y, y') = 0$

where
- x : The independent variable
- y : The dependent variable
- y' : The derivative dy/dx

- The 1st order ODE can be written as $\frac{dy}{dx} = f(x,y)$ or $y' = f(x,y)$

- Methods of Solution

① Separation of variables (separable equation) - Sec 1.3

$f(x,y) = g(x)\,h(y)$, $\frac{dy}{dx} = g(x)\,h(y)$; separable eq.

$$\Rightarrow \frac{dy}{h(y)} = g(x)\,dx \Rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx$$

Example

Solve $\frac{dy}{dx} = -y^2$ $g(x) = -1$, $h(y) = +y^2$ Sep

Sol.

$$\frac{dy}{+y^2} = -dx, \Rightarrow \int \frac{dy}{+y^2} = \int -dx \Rightarrow \boxed{\frac{1}{y} = x + c}$$

or $y = \frac{1}{x+c}$, c: constant (From Initial condition) (IC's)

Example Solve $y' = -2xy$, Initial condition $y(0) = 1.8$

↳ Initial value problem IVP

Sol $\frac{dy}{dx} = \underbrace{-2x}_{g(x)}\ \underbrace{y}_{h(y)}$

$$\Rightarrow \frac{dy}{y} = -2x\,dx,\ \int \frac{dy}{y} = \int -2x\,dx \Rightarrow \ln y = -x^2 + c_1, \text{ take } e^{(\ )}$$

$$\Rightarrow y = e^{-x^2+c_1} = e^{c_1} \cdot e^{-x^2} \Rightarrow y(x) = c\,e^{-x^2}$$

cont'd $y(x) = C e^{-x^2}$, C: constant (IC's)

$y(0) = 1.8 = C e^{-(0)} \Rightarrow C = 1.8$

$y(x) = 1.8\, e^{-x^2}$

Problem set 1.3: Solve: 2, 3, 6, 8, 10, 12, 15

2 Exact equations - Sec 1.4

$\frac{dy}{dx} = f(x,y)$, we can rewrite as:

$P(x,y)\,dx + Q(x,y)\,dy = 0$ "Differential form"

Example

$\frac{dy}{dx} = \frac{\sin y}{2y - x\cos y}$, $P(x,y) = \sin y$, $Q(x,y) = 2y - x\cos y$

$\Rightarrow \boxed{\sin y\, dx + (x\cos y - 2y)\,dy = 0}$ —

- The differential form can be re-written as:

$d\phi = P(x,y)\,dx + Q(x,y)\,dy$

The "total differential" or $\phi(x,y)$ is expressed as:

$d\phi = \frac{\partial \phi}{\partial x}\cdot dx + \frac{\partial \phi}{\partial y}\cdot dy$

$\partial\phi/\partial x = P(x,y)$

$\partial\phi/\partial y = Q(x,y)$

If $d\phi = 0$, then $\phi(x,y) = 0$ (Implicit solution)

- The differential form of the ODE is called "exact", If the following condition is satisfied:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Thus, For an exact ODE, then the Solution can be determined in it's implicit form by direct integration of the total differential $\phi(x,y)$

$$d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy$$

Example $\frac{dy}{dx} = \frac{-(3x^2 + y\cos x)}{\sin x - 4y^3}$

write in the differential form $P(x,y)dx + Q(x,y)dy = 0$

$\Rightarrow \underbrace{(3x^2 + y\cos x)}_{P(x,y)} dx + \underbrace{(\sin x - 4y^3)}_{Q(x,y)} dy = 0$

$\rightarrow$ Check if exact $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial y} = \cos x$ and $\frac{\partial Q}{\partial x} = \cos x \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ✓ exact!

Now, we can obtain the solution by integrating the total differential $d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy$

So, ① $\frac{\partial \phi}{\partial x} = P(x,y) = 3x^2 + y\cos x$, $d\phi = \frac{\partial \phi}{\partial x} \cdot dx \Rightarrow \phi = \int \frac{\partial \phi}{\partial x} \cdot dx$

$\Rightarrow \phi = \int (3x^2 + y\cos x)\, dx = \underline{x^3 + y\sin x} + k(y)$ — (1)

To find $k(y)$, $\frac{d\phi}{dy} = Q(x,y) = 0 + \cancel{\sin x} + \frac{dk}{dy} = \cancel{\sin x} - 4y^3$

$\Rightarrow \frac{dk}{dy} = -4y^3$, $k(y) = \int -4y^3 dy = -y^4 + C = k(y)$

$\Rightarrow$ then, $\phi = x^3 + y\sin x + k(y)$

$\Rightarrow \quad \phi = x^3 + y\sin x - y^4 = C$

For Implicit solution $\underline{d\phi = 0}$

or we can use (2) $\frac{\partial \phi}{\partial y} = Q(x,y) = \sin x - 4y^3$

thus, $\phi = \int (\sin x - 4y^3)\, dy = y \sin x - y^4 + K(x)$

To find $K(x)$, $\frac{d\phi}{dx} = P(x,y) = y\cos x - 0 + \frac{dK}{dx} = 3x^2 + y\cos x$

$\Rightarrow \frac{dK}{dx} = 3x^2$, $K(x) = \int 3x^2 dx = x^3 + C = K(x)$

then, $\boxed{\phi = y \sin x - y^4 + x^3 = C}$ ← Same as before!

(3) Integrating Factors - Sec 1.4

For 1st order ODE, the linear ODE is:

$a_0(x) y' + a_1(x) y = f(x)$, where $a_0(x) \neq 0$

we can rewrite ⇒ $y' + \underbrace{\frac{a_1(x)}{a_0(x)}}_{= p(x)} y = \underbrace{\frac{f(x)}{a_0(x)}}_{g(x)}$

$\Rightarrow y' + p(x) y = g(x)$, How to Solve?

we have two cases
(1) Homogeneous ODE $g(x) = 0$
(2) Non-Homogeneous ODE $g(x) \neq 0$

(1) Homogeneous case $g(x) = 0$

The ODE becomes $\frac{dy}{dx} + p(x) y = 0$ ($\frac{dy}{dx} \to y'$)

so, $\frac{dy}{dx} = -p(x) y \Rightarrow \frac{dy}{y} = -p(x)\, dx$, Integrate

$\ln y = \int -p(x) dx + C$, take $e^{(\ )}$

$\Rightarrow y(x) = A e^{\int -p(x) dx}$, $\ln A = C$ or $A = e^C$

2 Non- Homogeous case $g(x) \neq 0$ (Sec 1.5)

$y' + p(x)\,y = g(x)$ ← multiply by (σ) → Integrating factor

So, $\sigma y' + \sigma p(x)\, y = \sigma g(x)$ —(1)

Let $\frac{d}{dx}(\sigma y) = \sigma y' + \sigma p(x) y$ —(2)

So, $\frac{d}{dx}(\sigma y) = \sigma g(x)$ — (3)

Back to equation (2), $\cancel{\sigma y'} + \sigma' y = \cancel{\sigma y'} + \sigma p(x)\, y$

$\sigma' = \sigma p(x)$ or $\sigma'(x) = \sigma(x)\, p(x)$

$\frac{d\sigma}{dx} = \sigma(x)\, p(x)$ " we return to homog. case "

Thus, $\sigma(x) = e^{\int p(x)dx}$ ← Subst. in eq(3)

$\Rightarrow \frac{d}{dx}\left(e^{\int p(x)dx} \cdot y(x)\right) = e^{\int p(x)dx} \cdot g(x)$

or

$e^{\int p(x)dx} \cdot y(x) = \int e^{\int p(x)dx} \cdot g(x)\, dx + C$

$$y(x) = e^{-\int p(x)dx} \cdot \left[\int e^{\int p(x)dx}\, g(x)\, dx + C\right]$$

→ This is the general solution for 1^{st} ODE "linear"

$a_0(x)\, y' + a_1(x)\, y = f(x)$

or $y' + p(x)\, y = g(x)$

If $g(x) = 0$ " Homogen" $\Rightarrow y(x) = e^{-\int p(x)dx} \cdot C$

constant!

See example 1 page 29 textbook!

Practice Problems - Prob Set 1.5 Page 34
3, 5, 8, 10, 11, 13