Base Excitation (2.4)

$+\uparrow \sum F = m\ddot{x}$

$-c(\dot{x}-\dot{y}) - k(x-y) = m\ddot{x}$

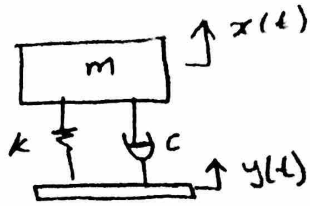
$\Rightarrow \quad m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$

$\Rightarrow \quad m\ddot{x} + c\dot{x} + kx = \underbrace{\boxed{c\dot{y} + ky}}_{F(t)}$

$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = 2\zeta w_n \dot{y} + w_n^2 y$

$x(t) = x_h + x_p(t)$

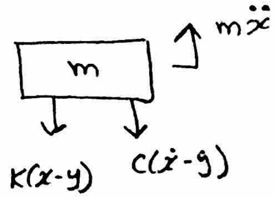
as before



$y(t) = Y \sin w_b t$

base Freq

F.B.D



$y(t) = Y \sin w_b t$

$\dot{y}(t) = w_b Y \cos w_b t$

$x_p \Rightarrow$

$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = \underbrace{\boxed{2\zeta w_n Y w_b \cos w_b t}}_{f_{o_1} \cos w_b t \,\to\, x_{p_1}(t)} + \underbrace{\boxed{w_n^2 Y \sin w_b t}}_{f_{o_2} \sin w_b t \,\to\, x_{p_2}(t)}$

From ODE's

$x_p(t) = x_{p_1}(t) + x_{p_2}(t)$

For $x_{p_1}(t)$, remember from last class $\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = f_o \cos wt$, $f_o = \frac{F_o}{m}$

$x_p(t) = X \cos(wt - \theta)$, $X = \dfrac{f_o}{\sqrt{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2}}$, $\theta = \tan^{-1}\left(\dfrac{2\zeta w w_n}{w_n^2 - w^2}\right)$

therefore

$x_{p_1}(t) = X_1 \cos(w_b t - \theta_1)$

where $X_1 = \dfrac{f_{o_1}}{\sqrt{(w_n^2 - w_b^2)^2 + (2\zeta w_b w_n)^2}} = \dfrac{2\zeta w_b w_n Y}{\sqrt{(w_n^2 - w_b^2)^2 + (2\zeta w_n w_b)^2}}$

$\theta_1 = \tan^{-1}\left(\dfrac{2\zeta w_n w_b}{w_n^2 - w_b^2}\right)$

For $x_{p_2}(t)$

$x_{p_2}(t) = \alpha_2 \cos\omega_b t + \beta_2 \sin\omega_b t$ and apply in

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 Y \sin\omega_b t$ to Find α_2 and β_2

then

$$x_{p_2}(t) = X_2 \sin(\omega_b t - \theta_1)$$

$$X_2 = \frac{\omega_n^2 Y}{\sqrt{(\omega_n^2-\omega_b^2)^2 + (2\zeta\omega_b\omega_n)^2}}, \quad \theta_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2-\omega_b^2}\right)$$

Finally,

See problem 2.48 for proof

$$x_p(t) = x_{p_1}(t) + x_{p_2}(t)$$

$$x_p(t) = \boxed{\omega_n Y \left[\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}\right]^{1/2}} \cos(\omega_b t - \theta_1 - \theta_2)$$

$$\theta_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2-\omega_b^2}\right), \quad \theta_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta\omega_b}\right)$$

or $$x_p(t) = X \cos(\omega_b t - \theta_1 - \theta_2)$$

$$X = Y \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2} \qquad r = \frac{\omega_b}{\omega_n}$$

$$\frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

we need the plots

displacement transmissibility (T_d)

→ Describes how motion is transferred from the base to the mass (system)

Transmissibility Plots

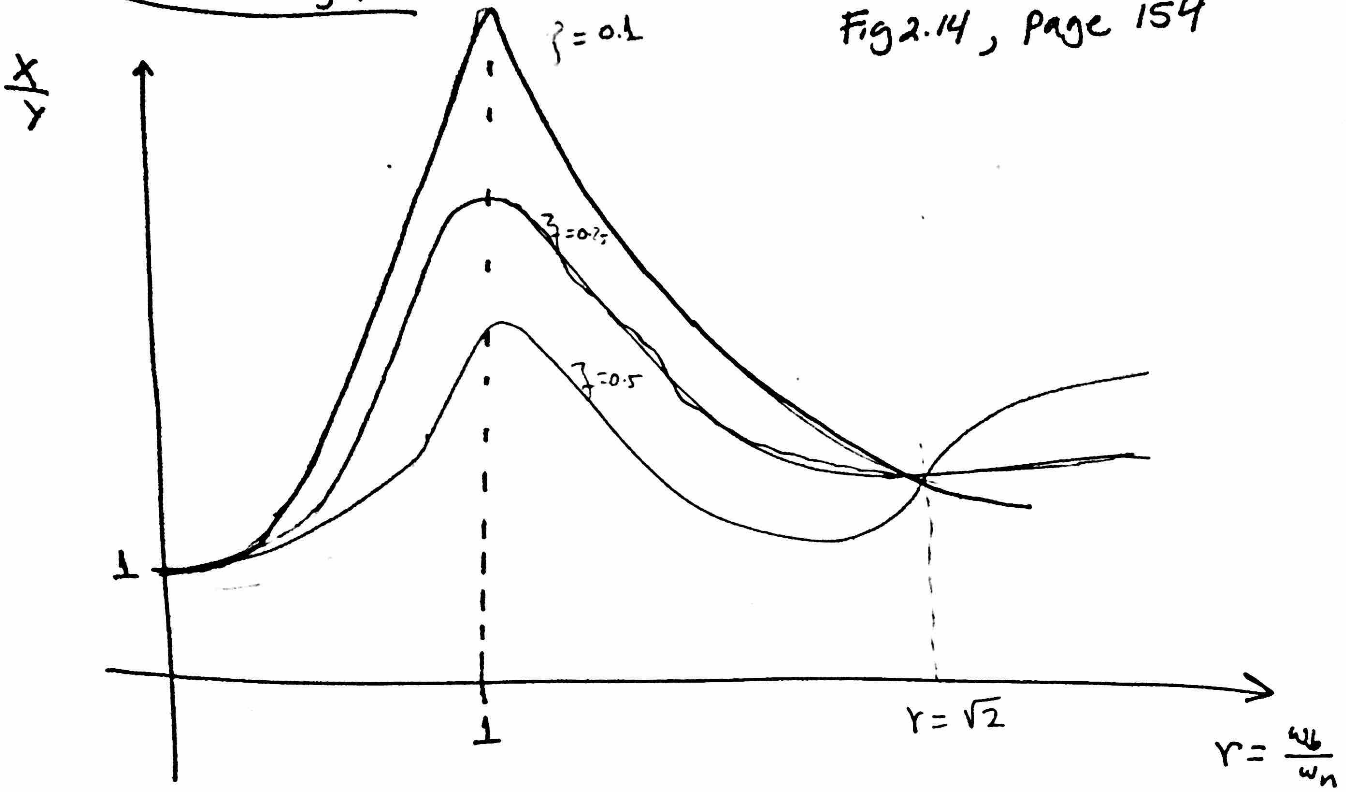


Fig 2.14, Page 154

* Back to the equation of motion

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x-y) = 0$$

$$c(\dot{x} - \dot{y}) + k(x-y) = \underbrace{-m\ddot{x}}_{\text{Force}} = F(t)$$

$$F(t) = -m\ddot{x}(t)$$

For steady-state sol

$$x(t) = X\cos(\omega_b t - \theta_1 - \theta_2)$$

derive twice and substitute in $F(t) = -m\ddot{x}$

$$\Rightarrow \quad F(t) = +\omega_b^2\, m X \cos(\omega_b t - \theta_1 - \theta_2)$$

$$r = \frac{\omega_b}{\omega_n} \Rightarrow r^2 = \frac{\omega_b^2}{\omega_n^2} \Rightarrow \omega_b^2 = r^2 \omega_n^2 = \frac{r^2 k}{m}$$

$$F(t) = \frac{r^2 k}{\cancel{m}} \cdot \cancel{m}\, X \cos(\omega_b t - \theta_1 - \theta_2)$$

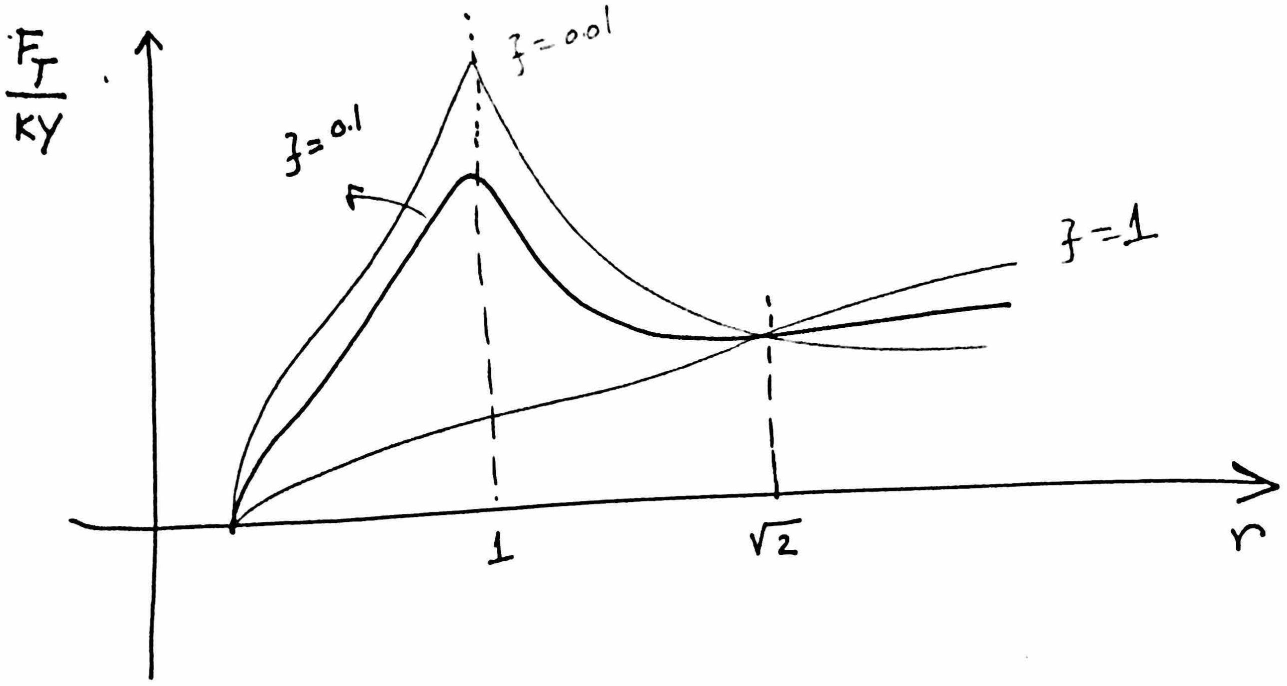
$$F_T = r^2 k X$$

$$F_T = k Y r^2 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

force transmissibility

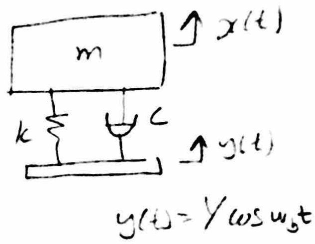
measures how base motion Y results in force amplitude applied $\Rightarrow$

$$\frac{F_T}{kY} = r^2 \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2} \Rightarrow \frac{F_T}{kY} = r^2\, \frac{X}{Y}$$



At resonance $(r=1)$

$$\frac{F_T}{kY} = \left[\frac{1+(2\zeta)^2}{(2\zeta)^2}\right]^{1/2}$$



Example: For a base Excitation problem
$m = 100\,kg$, $c = 30\,kg/s$
$k = 2000\,N/m$, $Y = 0.03\,m$, $\omega_b = 6\,rad/s$

Find ① X/Y ② F_T/kY

Solution

$$\frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}, \quad r = \frac{\omega_b}{\omega_n}$$

$$\omega_n = \sqrt{\frac{m}{k}} = \sqrt{\frac{100}{2000}} = 4.472\ rad/s \Rightarrow r = \frac{6}{4.472} = 1.342$$

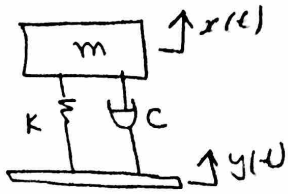
$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{30}{2\sqrt{(100)(2000)}} = 0.034$$

$$\Rightarrow \frac{X}{Y} = \left[\frac{1+(2(0.034)(1.342))^2}{(1-1.342)^2+(2(0.034)(1.342))^2}\right]^{1/2} \Rightarrow \frac{X}{Y} = 0.557$$

$$\frac{F_T}{kY} = r^2\,\frac{X}{Y} = (1.342)^2(0.557) = 1.003$$

* Relative motion

$y(t) = Y \cos\omega_b t$



EOM

$$m\ddot{x} + c(\dot{x}-\dot{y}) + K(x-y) = 0$$

Define relative motion $z(t)$

$$\left.\begin{aligned} z(t) &= x(t) - y(t) \\ \dot{z}(t) &= \dot{x}(t) - \dot{y}(t) \\ \ddot{z}(t) &= \ddot{x}(t) - \ddot{y}(t) \end{aligned}\right\} \text{subs in EOM}$$

$z(t)$: Relative motion describes the mass motion only.

$x(t)$: Absolute motion describes the total motion of the mass and the base

$$\Rightarrow \quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \qquad \div m$$

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y}$$

$\ddot{y} = -\omega_b^2 Y \cos\omega_b t$

Solution

$$z(t) = \underbrace{z_h(t)}_{\text{Homog. as in Free Vib}} + \underbrace{z_p(t)}_{\text{particular}}$$

$$\left.\begin{aligned} z_p(t) &= \alpha\cos\omega_b t + \beta\sin\omega_b t \\ \dot{z}_p(t) &= -\omega_b\alpha\sin\omega_b t + \omega_b\beta\cos\omega_b t \\ \ddot{z}_p(t) &= -\omega_b^2\alpha\cos\omega_b t - \omega_b^2\beta\sin\omega_b t \end{aligned}\right\}$$

$$\left(-\omega_b^2\alpha\cos\omega_b t - \omega_b^2\beta\sin\omega_b t\right) + 2\zeta\omega_n\left(-\omega_b\alpha\sin\omega_b t + \omega_b\beta\cos\omega_b t\right)$$

$$+ \omega_b^2\left(\alpha\cos\omega_b t + \beta\sin\omega_b t\right) = -\omega_b^2 Y\cos\omega_b t$$

$\cos\omega_b t$

$$\cancel{\cos\omega_b t}\left(-\omega_b^2\alpha + 2\zeta\omega_n\omega_b\beta + \omega_n^2\alpha\right) = -\omega_b Y \cancel{\cos\omega_b t}$$

$$\cancel{\sin\omega_b t}\left(-\omega_b^2\beta - 2\zeta\omega_n\omega_b\alpha + \omega_n^2\beta\right) = 0$$

$$\Rightarrow \quad \alpha = \frac{-\omega_b Y(\omega_n^2 - \omega_b^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega_b)^2}, \quad \beta = \frac{-2\zeta\omega_n\omega_b Y}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega_b)^2}$$

Cont'd

$$z(t) = Z\cos(\omega_b t - \gamma)$$

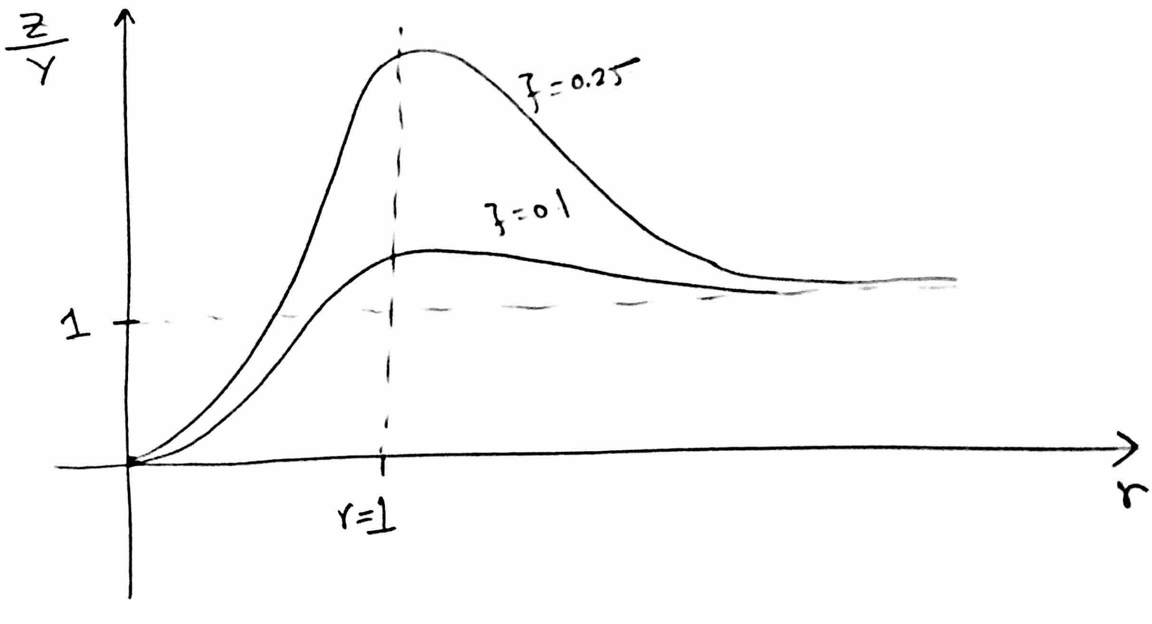
$$Z = \frac{\omega_b^2 Y}{\sqrt{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}}\quad,\quad \gamma = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2-\omega_b^2}\right)$$

$$\frac{Z}{Y} = \frac{\omega_b^2}{\sqrt{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$\rightarrow 1$

$r \longrightarrow \infty$

$$\gamma = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$



* Base Excitation

* In this class we're going to find the total solution for a base excitation problem

we will consider both

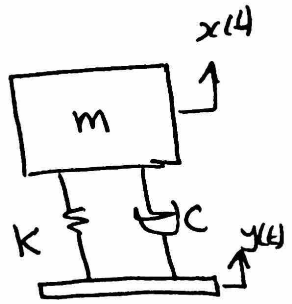
→ Absolute motion $[x(t)]$

→ Relative motion $[z(t)]$

* As we know, total solution consists of two parts

① Homogeneous

② particular (non-homog.)



- For Absolute motion

$$\underbrace{x(t)}_{\text{total sol.}} = \underbrace{x_h(t)}_{\text{homog. sol.}} + \underbrace{x_p(t)}_{\text{particular sol.}}$$

- For relative motion

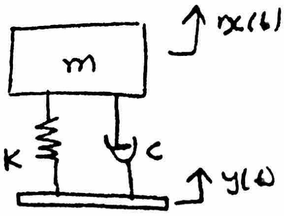
$$\underbrace{z(t)}_{\text{total}} = \underbrace{z_h(t)}_{\text{homog.}} + \underbrace{z_p(t)}_{\text{particular}}$$

$x_p(t)$ and $z_p(t)$, we found them in class

for $y(t) = Y \cos \omega_b t$

Base Excitation, cont'd

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y$$ ← Equation of motion



$$x(t) = x_h(t) + x_p(t)$$

$$x_p(t) = X\cos(\omega_b t - \theta_1 - \theta_2)$$

$$X = Y\left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}, \quad \theta_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right), \quad \theta_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta\omega_b}\right)$$

$x_h(t)$?

$\zeta < 1$ (under damped)

$$x_h(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t), \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + X\cos(\omega_b t - \theta_1 - \theta_2)$$

Initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$

⇒ we find A and B

It is important to understand how to find constants A and B

$\zeta > 1$ overdamped

$$x_h(t) = e^{-\zeta\omega_n t}\left(A e^{+\omega_n\sqrt{\zeta^2-1}\,t} + B e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

$$x(t) = e^{-\zeta\omega_n t}\left(A e^{\omega_n\sqrt{\zeta^2-1}\,t} + B e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right) + X\cos(\omega_b t - \theta_1 - \theta_2)$$

Initial conditions, $x(0) = x_0$, $\dot{x}(0) = v_0$

⇒ we can find A and B

$\zeta = 1$ Critical damping

$$x_h(t) = A e^{-\omega_n t} + Bt\, e^{-\omega_n t}$$

$$x(t) = A e^{-\omega_n t} + Bt\, e^{-\omega_n t} + X\cos(\omega_b t - \theta_1 - \theta_2)$$

IC's $x(0) = x_0$, $\dot{x}(0) = v_0$

⇒ find A and B

- Relative motion

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n x = 2\zeta\omega_n\dot{y} + \omega_n^2 y$

Using relative motion $z(t) = x(t) - y(t)$ substitute

$\Rightarrow \ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n z = -\ddot{y}$

$z(t) = z_h(t) + z_p(t)$

$z_p(t) = Z\cos(\omega_b t - \gamma)$ $\Big\}$ Last class

z_h? (Now, find homog. ~~solution~~ and total Solution)

$\zeta < 1$ underdamped

$z_h(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$

$z(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + Z\cos(\omega_b t - \gamma)$

Initial condition $x(0) = x_0$, $\dot{x}(0) = v_0$

$z(0) = x(0) - y(0) \Rightarrow z(0) = x_0 - y(0)$

$\dot{z}(0) = \dot{x}(0) - \dot{y}(0) \Rightarrow \dot{z}(0) = v_0 - \dot{y}(0)$ $\Big\}$ then Find A and B

$\zeta > 1$

$z_h(t) = e^{-\zeta\omega_n t}\left(A e^{\omega_n\sqrt{\zeta^2-1}\,t} + B e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$

$z(t) = e^{-\zeta\omega_n t}\left(A e^{\omega_n\sqrt{\zeta^2-1}\,t} + B e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right) + Z\cos(\omega_b t - \gamma)$

as before from IC's Find A and B

$\zeta = 1$

$z_h(t) = A e^{-\omega_n t} + B t e^{-\omega_n t}$

$z(t) = A e^{-\omega_n t} + B t e^{-\omega_n t} + Z\cos(\omega_b t - \gamma)$

as before, from IC's Find A and B.