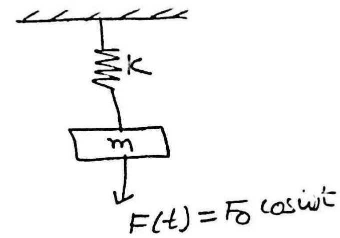
# * Forced vibration of undamped System
## (Harmonic Forces Excitation)

$F_0$ = Force amplitude (N)
$\omega$ = Excitation Frequency (rad/s)




Equation of motion

$$m\ddot{x} + kx = F_0 \cos \omega t$$

$$\ddot{x} + \omega_n^2 x = f_0 \cos \omega t \quad , \quad f_0 = \frac{F_0}{m}$$

Solution

$$x(t) = \underbrace{x_h(t)}_{\text{Homog.}} + \underbrace{x_p(t)}_{\text{Particular}}$$

we know, (From Ch.1) $\rightarrow \ddot{x} + \omega_n^2 x = 0$

$$x_h(t) = A \cos \omega_n t + B \sin \omega_n t$$

For $x_p(t)$

$$x_p(t) = \alpha \cos \omega t + \beta \sin \omega t$$

Substitute in EOM

$$x_p(t) = \alpha \cos \omega t + \beta \sin \omega t$$
$$\dot{x}_p(t) = -\omega \alpha \sin \omega t + \omega \beta \cos \omega t$$
$$\ddot{x}_p(t) = -\omega^2 \alpha \cos \omega t - \omega^2 \beta \sin \omega t$$

$$-\omega^2 \alpha \cos \omega t - \omega^2 \beta \sin \omega t + \omega_n^2 \alpha \cos \omega t + \omega_n^2 \beta \sin \omega t = f_0 \cos \omega t$$

terms with $\cos \omega t$

$$-\omega^2 \alpha \cancel{\cos \omega t} + \omega_n^2 \alpha \cancel{\cos \omega t} = f_0 \cancel{\cos \omega t}$$

$$\Rightarrow \boxed{\alpha = \frac{f_0}{\omega_n^2 - \omega^2}}$$

terms with $\sin \omega t$

$$-\omega^2 \beta \cancel{\sin \omega t} + \omega_n^2 \beta \cancel{\sin \omega t} = 0$$

$$\Rightarrow \boxed{\beta = 0}$$

$$\Rightarrow x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

Total solution

$$x(t) = A \cos \omega_n t + B \sin \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

A and B constants from Initial conditions.

## For initial conditions

$x(0) = x_0$ and $\dot{x}(0) = v_0 \Rightarrow$ Find $A$ and $B$

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{f_0}{\omega_n^2-\omega^2}\cdot\cos\omega t$$

$$x(0) = x_0 = A + 0 + \frac{f_0}{\omega_n^2-\omega^2}$$

$$\Rightarrow A = x_0 - \frac{f_0}{\omega_n^2-\omega^2}, \quad f_0 = F_0/m$$

$\dot{x}(0) = v_0 \Rightarrow$

$$\dot{x}(t) = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t - \omega\frac{f_0}{\omega_n^2-\omega^2}\sin\omega t$$

$$\dot{x}(0) = v_0 = 0 + \omega_n B - 0$$

$$\Rightarrow B = \frac{v_0}{\omega_n}$$

$$\Rightarrow x(t) = \left(x_0 - \frac{f_0}{\omega_n^2-\omega^2}\right)\cos\omega_n t + \frac{v_0}{\omega_n}\cdot\sin\omega_n t + \frac{f_0}{\omega_n^2-\omega^2}\cos\omega t$$

$$\omega_n = \sqrt{\frac{k}{m}}, \quad f_0 = F_0/m$$

* **If** $\omega_n = \omega$, Natural Freq = Excitation Freq.

$$x(t) \longrightarrow \infty$$

(Resonance)

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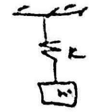
* **If** $F(t) = F_0\cos\omega t = 0$ $\rightarrow F_0 = 0$

$$x(t) = x_0\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

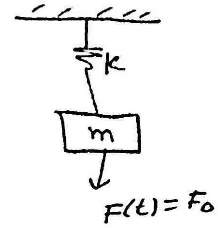
Back to free-undamped system



* Constant Force, $F(t) = F_0$ , $\omega = 0$

$x(0) = x_0$

$\dot{x}(0) = v_0$




Equation of motion

$$\ddot{x} + \omega_n x = f_0 \quad , \quad f_0 = \frac{F_0}{m}$$

$$x(t) = \left(x_0 - \frac{f_0}{\omega_n^2}\right)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t + \frac{f_0}{\omega_n^2}$$

Another Form: Find $x(t)$ , If $F(t) = F_0 \sin\omega t$

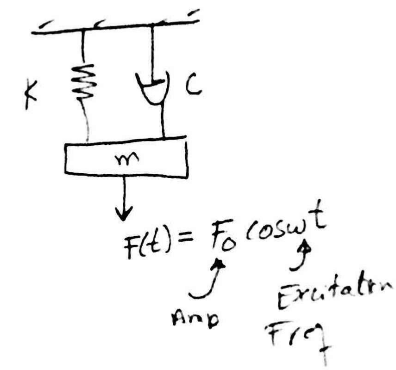
to practice

$$F(t) = F_0 \sin\omega t$$

$\rightarrow$ practice

# 2.2 Harmonic Excitation of Damped Systems

Equation of motion




$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos\omega t$$

$$2\zeta\omega_n = \frac{c}{m}, \quad \omega_n^2 = \frac{k}{m}, \quad f_0 = \frac{F_0}{m}$$

Solution

$$x(t) = \underbrace{x_h(t)}_{\text{Homog.}} + \underbrace{x_p(t)}_{\text{particular}}$$

$$x_p(t) = \alpha\cos\omega t + \beta\sin\omega t$$

$$\dot{x}_p(t) = -\omega\alpha\sin\omega t + \omega\beta\cos\omega t$$

$$\ddot{x}_p(t) = -\omega^2\alpha\cos\omega t - \omega^2\beta\sin\omega t$$

Substitute in governing equation

$$-\omega^2\alpha\cos\omega t - \omega^2\beta\sin\omega t + 2\zeta\omega_n(-\omega\alpha\sin\omega t + \omega\beta\cos\omega t)$$
$$+ \omega_n^2(\alpha\cos\omega t + \beta\sin\omega t) = f_0\cos\omega t$$

Collect

$$(-\omega^2\alpha + 2\zeta\omega\omega_n\beta + \omega_n^2\alpha)\cancel{\cos\omega t} = f_0\cancel{\cos\omega t}$$
$$(-\omega^2\beta - 2\zeta\omega\omega_n\alpha + \omega_n^2\beta)\cancel{\sin\omega t} = 0$$

$$(\omega_n^2 - \omega^2)\alpha + 2\zeta\omega\omega_n\beta = f_0$$
$$-2\zeta\omega\omega_n\alpha + (\omega_n^2 - \omega^2)\beta = 0$$

$$\begin{bmatrix} \omega_n^2 - \omega^2 & 2\zeta\omega\omega_n \\ -2\zeta\omega\omega_n & \omega_n^2 - \omega^2 \end{bmatrix}\begin{Bmatrix}\alpha \\ \beta\end{Bmatrix} = \begin{Bmatrix} f_0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad \alpha = \frac{(\omega_n^2 - \omega^2)f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

$$\beta = \frac{2\zeta\omega\omega_n f_0}{(\omega_n^2-\omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

$$x_p(t) = \alpha\cos\omega t + \beta\sin\omega t$$

Cont'd

We can write the $x_p(t)$, as:

$$x_p(t) = X\cos(\omega t - \theta)$$

where $X = \sqrt{\alpha^2 + \beta^2}$ , $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}} \quad , \quad \theta = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$

Then total solution

$$x(t) = x_h(t) + x_p(t)$$

$x_h$: $\zeta < 1$ , $\zeta = 1$ , $\zeta > 1$ , remember $\zeta > 0$ , If $\zeta = 0$ (undamped system)

* $x_h(t)$ is called transient solution, $x_h \to 0$ when $t \to \infty$

* $x_p(t)$ is called steady-state solution, $x_p \neq 0$ for any $t$

Because $x_h \to 0$ when $t \to \infty$, we focus on the steady state response (solution)

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}} \quad , \quad \theta = \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$

remember $f_0 = F_0/m$ and take $\omega_n^2$ outside ($* \frac{\omega_n^2}{f_0}$)

and $\theta \Rightarrow \frac{1/\omega_n^2}{1/\omega_n^2}$

$$\Rightarrow \frac{X\omega_n^2}{f_0} = \frac{XK}{F_0} = \frac{1}{\sqrt{(1-r^2) + (2\zeta r)^2}}$$

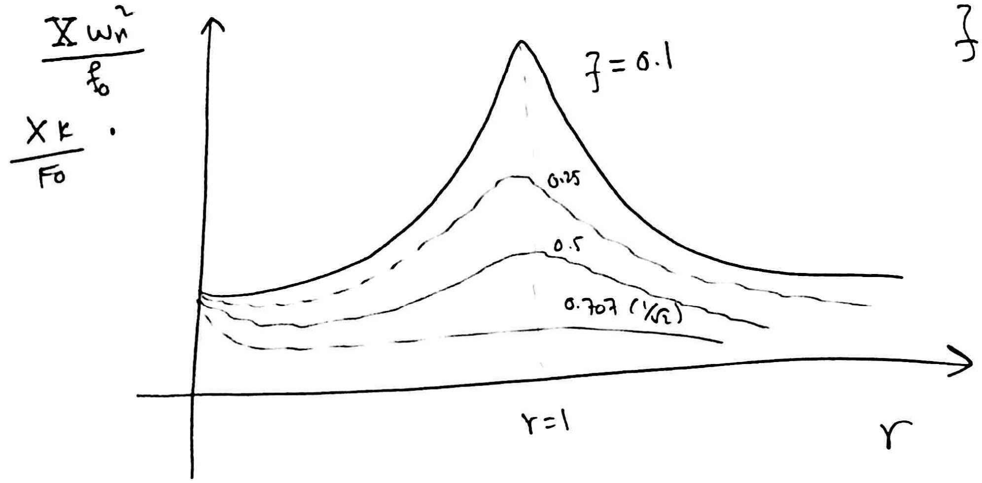
(Normalized force magnitude)

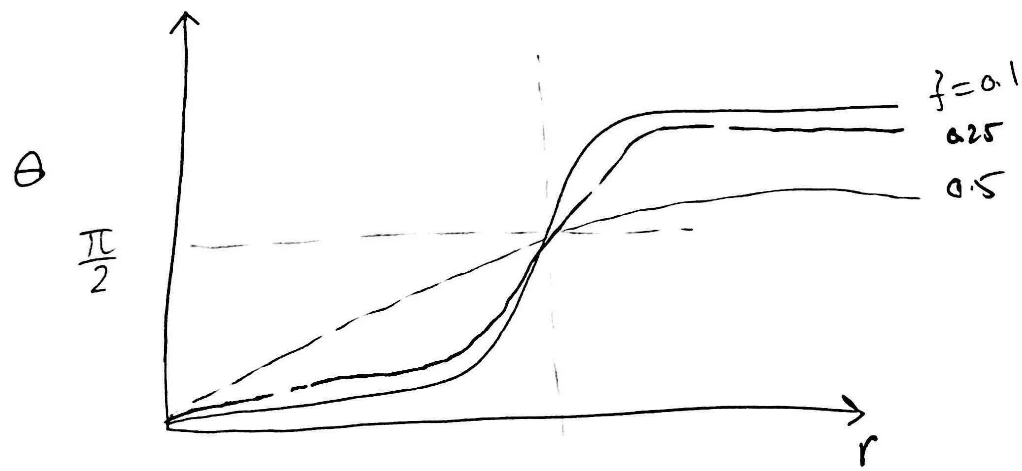
$$\Rightarrow \theta = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \quad , \quad r = \frac{\omega}{\omega_n}$$

Freq ratio

If $\omega = \omega_n$ or $r = 1$ $\Rightarrow$ resonance

$\zeta \downarrow$ peak sharpness $\uparrow$



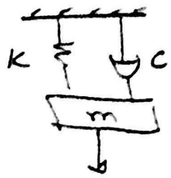




$\zeta \uparrow$ linear $\uparrow$

* Underdamped system

$$m\ddot{x} + c\dot{x} + Kx = F_0 \cos\omega t$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos\omega t, \quad f_0 = F_0/m$$





$F(t) = F_0 \cos\omega t$

Solution

$$x(t) = x_h(t) + x_p(t)$$

If $\zeta < 1$ (underdamped)

$$x_h(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_n t + B\sin\omega_n t\right)$$

$$x_p(t) = \alpha\cos\omega t + \beta\sin\omega t$$

$$\alpha = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

$$\beta = \frac{2\zeta\omega\omega_n f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

total solution

$$x(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_n t + B\sin\omega_n t\right) + \alpha\cos\omega t + \beta\sin\omega t$$

A and B constants from Initial conditions.

$x(0) = x_0$, $\dot{x}(0) = v_0$ $\Rightarrow$ Find A and B

$$A = x_0 - \frac{f_0(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$B = \frac{\zeta\omega_n}{\omega_d}\left(x_0 - \frac{f_0(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\right) - \left(\frac{2\zeta\omega^2\omega_n}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\right)\cdot\frac{1}{\omega_d} + \frac{v_0}{\omega_d}$$

$$B = \frac{\zeta\omega_n}{\omega_d} A + \frac{v_0}{\omega_d} - \frac{\omega\beta}{\omega_d}$$

Ask them to do the same for $\zeta = 1$
$\zeta > 1$

combine homogeneous and particular solutions to obtain total solution $x(0) = x_0$, $\dot{x}(0) = v_0$