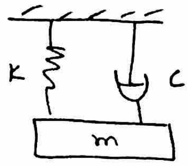
1.3 Viscous Damping
(Free vibration of Damped systems)

Spring - mass - damper system

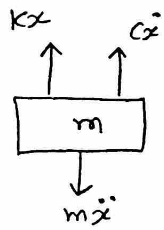
C ⟹ - Damping element } Viscous Damping
- Dash pot } (oil)



Damping factor (constant) ⟹ C [N.s/m] or [kg/s]

Forces from C ⟹ $C\dot{x}$, like $K \Rightarrow Kx$

Equation of motion



Dynamic

$$+\uparrow \sum F = -m\ddot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = 0$$

Damped - Free SDOF system

Divide by m

$$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = 0$$

→ zeta (unitless)

ζ = damping ratio (we measure this)

$$2\zeta w_n = \frac{C}{m} \Rightarrow \zeta = \frac{C}{2 w_n m}$$

remember $w_n = \sqrt{\frac{k}{m}} \Rightarrow \zeta = \frac{C}{2\sqrt{km}}$

$\zeta = \frac{C}{C_{cr}}$ → Critical damping coefficient

To solve let $x(t) = Ae^{\lambda t}$

$\dot{x}(t) = \lambda A e^{\lambda t}$, $\ddot{x} = \lambda^2 A e^{\lambda t}$

$$\lambda^2 \cancel{A e^{\lambda t}} + 2\zeta w_n \lambda \cancel{A e^{\lambda t}} + w_n^2 \cancel{A e^{\lambda t}} = 0$$

$$\lambda^2 + 2\zeta w_n \lambda + w_n^2 = 0$$

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_{1,2} = \frac{-2\zeta w_n \pm \sqrt{(2\zeta w_n)^2 - 4w_n^2}}{2}$$

$$\lambda_{1,2} = \frac{-2\zeta w_n \pm 2w_n\sqrt{\zeta^2 - 1}}{2} \Rightarrow \lambda_{1,2} = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$$

Cases

Most common ← ① $\zeta < 1$ ⟹ Complex roots (underdamped system)

$$\lambda_{1,2} = -\zeta w_n \pm \hat{i}\, w_n \sqrt{1 - \zeta^2}$$

② $\zeta = 1$ ⟹ Repeated root (Critical damping)

$$\lambda_{1,2} = -w_n$$

③ $\zeta > 1$ ⟹ Two real roots (overdamped)

$$\lambda_{1,2} = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1}$$

Undamped system $\zeta < 1$ $\lambda_{1,2} = -\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2}$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$x(t) = Ae^{\lambda t} \Rightarrow x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

$$x(t) = Ae^{-\zeta\omega_n t}\cdot e^{i\omega_n\sqrt{1-\zeta^2}\,t} + Be^{-\zeta\omega_n t}\cdot e^{-i\omega_n\sqrt{1-\zeta^2}\,t}, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$: damped Nat. Freq (rad/s)

$$\Rightarrow x(t) = e^{-\zeta\omega_n t}\left(Ae^{i\omega_d t} + Be^{-i\omega_d t}\right)$$

But $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ $\Rightarrow$ Euler's Equation

$$\Rightarrow x(t) = e^{-\zeta\omega_n t}\left[A(\cos\omega_d t + i\sin\omega_d t) + B(\cos\omega_d t - i\sin\omega_d t)\right]$$

$$= e^{-\zeta\omega_n t}\left[\underbrace{(A+B)}_{\alpha \text{ (Alpha)}}\cos\omega_d t + \underbrace{i(A-B)}_{\beta \text{ (Beta)}}\sin\omega_d t\right]$$

$$\Rightarrow x(t) = e^{-\zeta\omega_n t}\left(\alpha\cos\omega_d t + \beta\sin\omega_d t\right),$$ α, β Constant From Initial conditions.

IC's $x(0) = x_0$, $\dot{x}(0) = v_0 \Rightarrow \alpha, \beta$?

$x(0) = x_0 \Rightarrow x_0 = \alpha \Rightarrow \alpha = x_0$

$$\dot{x}(t) = e^{-\zeta\omega_n t}\left[-\omega_d\,\alpha\sin\omega_d t + \omega_d\,\beta\cos\omega_d t\right] - \zeta\omega_n e^{-\zeta\omega_n t}\left[\alpha\cos\omega_d t + \beta\sin\omega_d t\right]$$

$$\dot{x}(0) = v_0 = \omega_d\beta - \zeta\omega_n\underbrace{\alpha}_{=x_0}$$

$$\Rightarrow \beta = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

Another Solution form $x(t) = X\sin(\omega_d t + \phi)$ (X: Amplitude, ϕ: phase)

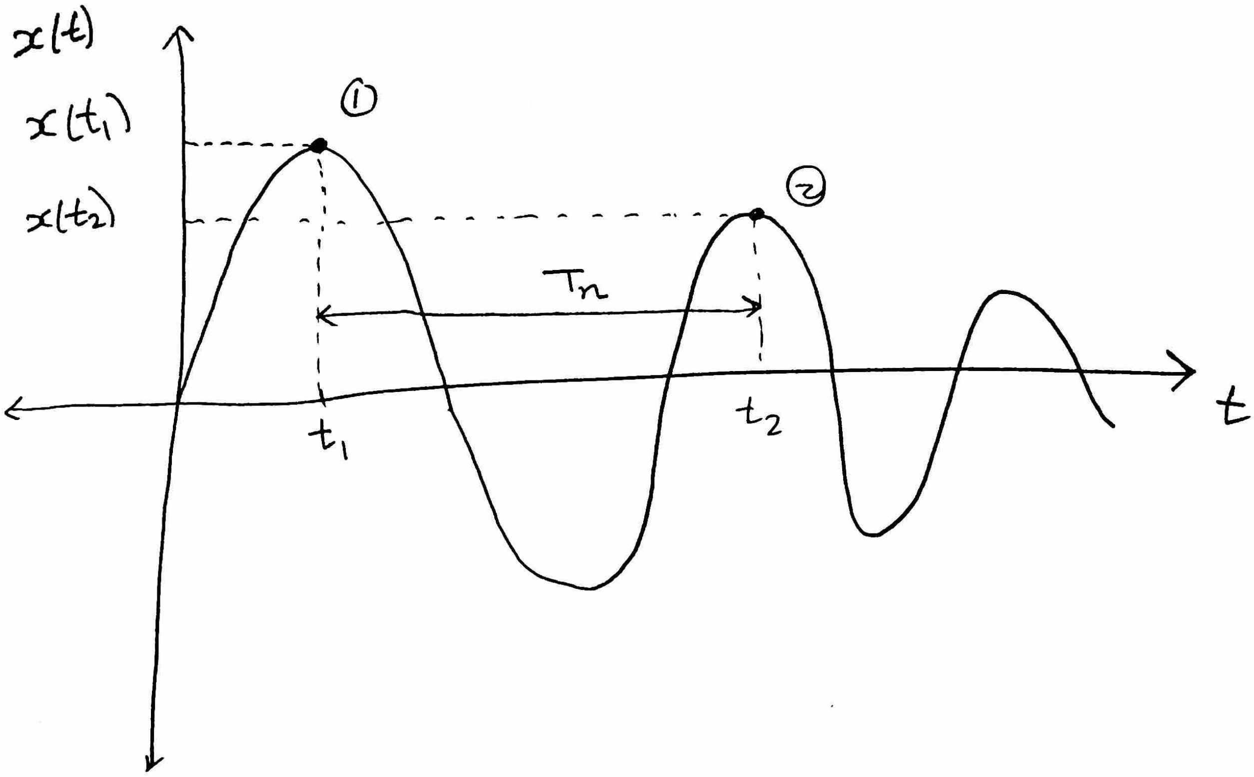
Remember $X\sin(\omega_d t + \phi) = X\cos\omega_d t\cdot\sin\phi + X\sin\omega_d t\cos\phi$

$$\Rightarrow X = \sqrt{(x_0)^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\,\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

= Let them see Figure 1.10 page 26 - textbook! "

plot $x(t)$, $\zeta < 1$



$$\ln \frac{x(t_1)}{x(t_2)} = \delta$$, δ: Log decrement

$$\Rightarrow \zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

Overdamped system $\zeta > 1$, $\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

$$x(t) = e^{-\zeta\omega_n t}\left[A e^{\omega_n\sqrt{\zeta^2-1}\,t} + B e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right]$$, A,B – From IC's

IC's $x(0) = x_0$, $\dot{x}(0) = v_0$

$$A = \frac{+v_0 + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$B = \frac{-v_0 + \left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

See Fig 1.11 page 27 For plots.

(3) Critical damping $\zeta = 1$, $\lambda_{1,2} = -\omega_n$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

$$x(t) = A e^{\lambda t} + Bt\, e^{\lambda t}$$ A, B From IC's

$$\Rightarrow x(t) = (A + Bt)\, e^{-\omega_n t}$$

IC's $x(0) = x_0$, $\dot{x}(0) = v_0$

$\Rightarrow$ $A = x_0$

$B = v_0 + \omega_n x_0$

ple $m = 49.2\ g$, $k = 857.8\ N/m$, $C = 0.11\ kg/s$

Find ζ

$$\zeta = \frac{C}{2m\omega_n} = \frac{C}{2\sqrt{km}} = \frac{0.11}{2\sqrt{(857.8)(49.2\times10^{-3})}} \Rightarrow \zeta = 0.0085$$

Example for system with equation of motion and If $m = 1\ kg$

$\ddot{x} + 150\dot{x} + 25x = 0$

Find ① K
② C
③ ζ

Solution

General Form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Compare $\Rightarrow \omega_n^2 = 25 = \frac{K}{m} \Rightarrow 25 = \left(\frac{K}{m}\right) \Rightarrow K = 25\ N/m$

$\Rightarrow 2\zeta\omega_n = 150 \Rightarrow \zeta = \frac{150}{2\omega_n} = \frac{150}{2\sqrt{25}} = 15$

$\Rightarrow \frac{C}{m} = 150 \Rightarrow C = 150 \times m \Rightarrow C = 150\ kg/s$

Example 1.3.2 page 29
(No plot), Just Calculations.