Chapter Two: Vibration of Discrete Systems: Brief review

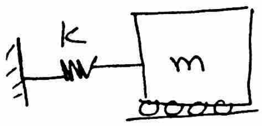
2.1 Vibration of Single-degree-of-freedom (SDOF) systems

2.1.1 Free vibrations (No damping, No Forces)

Initial conditions

$x(0) = x_0$, ~~$v(0) = v_0$~~ $\dot{x}(0) = v_0$

Equation of motion (EOM)



$$m\ddot{x} + kx = 0$$

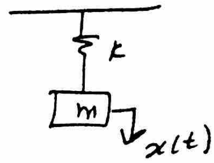
$$\ddot{x} + \frac{k}{m}x = 0 \qquad \sqrt{\frac{k}{m}} = \omega_n \quad \text{natural Frequency (rad/sec)}$$

m: mass
k: Spring constants

$$\ddot{x} + \omega_n^2 x = 0$$

Undamped vibration

$\ddot{x} + \omega_n^2 x = 0$, 2nd order Differential Equation (ODE)



Solution , we need to find $x(t)$

$x(t) = A e^{\lambda t}$

Velocity ← $\dot{x}(t) = \lambda A e^{\lambda t}$

Accel ← $\ddot{x}(t) = \lambda^2 A e^{\lambda t}$

$\lambda^2 A \cancel{e^{\lambda t}} + \omega_n^2 A \cancel{e^{\lambda t}} = 0 \Rightarrow \lambda^2 + \omega_n^2 = 0$

$\lambda^2 = -\omega_n^2 \Rightarrow \lambda_{1,2} = \pm \hat{i}\,\omega_n$, $\hat{i} = \sqrt{-1}$ complex number

⇒ From ODE

$x(t) = A_1 e^{\hat{i}\omega_n t} + A_2 e^{-\hat{i}\omega_n t}$

or

$x(t) = A \cos\omega_n t + B \sin\omega_n t$ ⇒ A and B constants ⇒ From initial conditions (IC's)

or

$x(t) = X \sin(\omega_n t + \phi)$

X : Vibration Amplitude
ϕ : phase shift (angle) } From IC's

How?

ID $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\Rightarrow \sin(\omega_n t + \phi) = \sin\omega_n t \cos\phi + \cos\omega_n t \sin\phi$

$X \sin(\omega_n t + \phi) = X \sin\omega_n t \cos\phi + X \cos\omega_n t \sin\phi$

compare to $x(t) = A\cos\omega_n t + B\sin\omega_n t$

$\frac{A}{B} \Leftarrow \begin{array}{l} A = X \sin\phi \\ B = X \cos\phi \end{array} \Rightarrow \begin{array}{l} A^2 = X^2 \sin^2\phi \\ B^2 = X^2 \cos^2\phi \end{array} +$

$A^2 + B^2 = X^2(\underbrace{\sin^2\phi + \cos^2\phi}_{=1})$

$X = \sqrt{A^2 + B^2}$

$\frac{\sin\phi}{\cos\phi} = \tan\phi = \frac{A}{B}$

$\phi = \tan^{-1}\left(\frac{A}{B}\right)$

If the Initial conditions are

$t=0$ Initial State

$x(0) = x_0$
$\dot{x}(0) = v_0$, what are A, B, X and ϕ?

Consider $x(t) = A\cos\omega_n t + B\sin\omega_n t$

Substitute IC's in $x(t)$

$$x(0) = x_0 = A + 0 \Rightarrow \boxed{A = x_0}$$

$$\dot{x}(t) = -\omega_n A\sin\omega t + \omega_n B\cos\omega_n t$$

$$\dot{x}(0) = v_0 = \omega_n B \Rightarrow \boxed{B = \frac{v_0}{\omega_n}}$$

$$\Rightarrow X = \sqrt{A^2+B^2} \Rightarrow \boxed{X = \sqrt{(x_0)^2 + \left(\frac{v_0}{\omega_n}\right)^2}}$$

$$\phi = \tan^{-1}\left(\frac{A}{B}\right) \Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{x_0}{v_0}\omega_n\right)}$$

plot $x(t) = X\sin(\omega_n t + \phi)$

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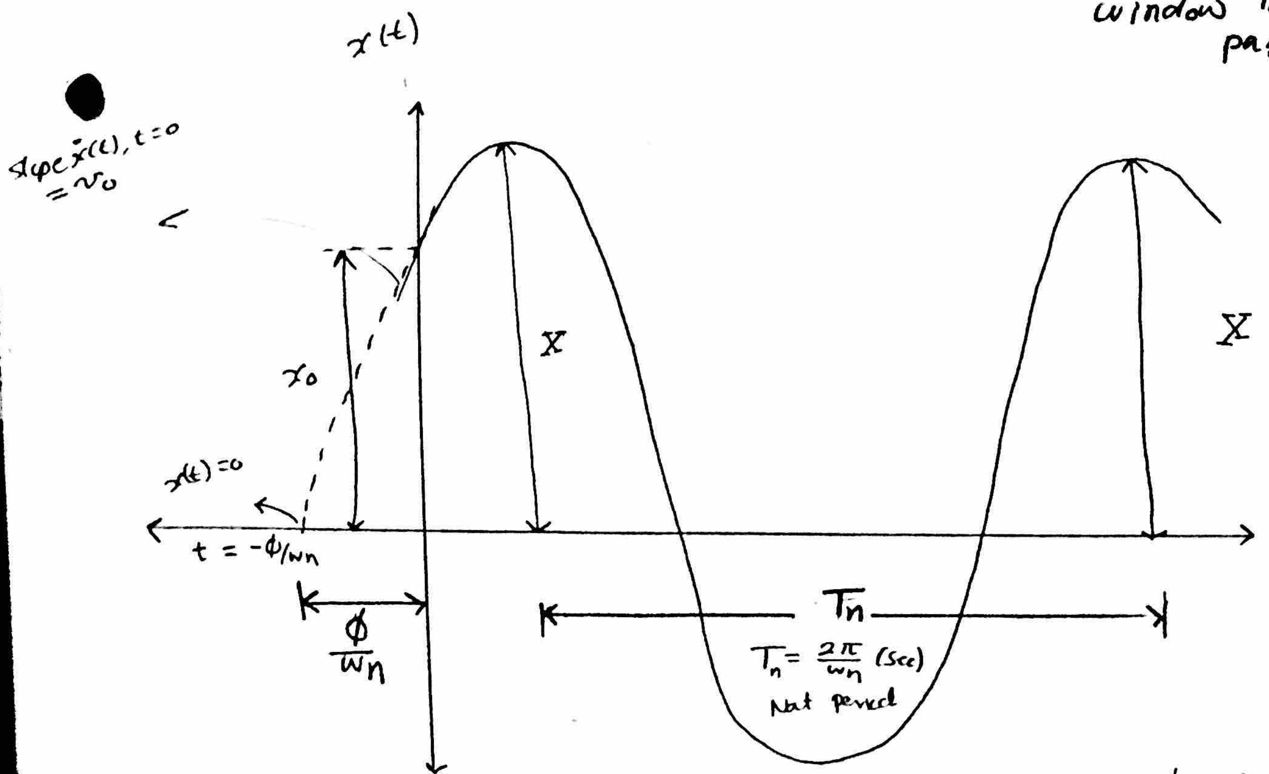


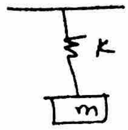
X = Amplitude

$$x(t) = 0 = \overset{\neq 0}{X}\sin(\omega_n t + \phi) \Rightarrow \sin(\omega_n t + \phi) = 0 \Rightarrow \omega_n t + \phi = 0$$
$$\Rightarrow t = \frac{-\phi}{\omega_n}$$

Nat frequency $(f_n) = 2\pi\omega_n = \frac{1}{T_n}$ (Hz) ample 1

$m = 30\ kg,\ f_n = 10\ Hz$

$k = ?$



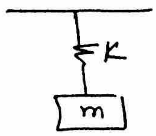
Solution

$$w_n = 2\pi f_n = \sqrt{\frac{k}{m}} \Rightarrow (2\pi f_n)^2 = \frac{k}{m} \Rightarrow k = m(2\pi f_n)^2$$

$$k = (30)(2\pi(10))^2 \Rightarrow k = 1.1184 \times 10^5\ N/m$$

Example 2

$m = 2\ kg,\ k = 200\ N/m$



find X and ϕ

(a) $x_0 = 2\ mm,\ v_0 = 1\ mm/s$

(b) $x_0 = -2\ mm,\ v_0 = 1\ mm/s$

Solution

$$X = \sqrt{x_0^2 + \left(\frac{v_0}{w_n}\right)^2} \quad , w_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{2}} \Rightarrow w_n = 10\ rad/s.$$

$$\phi = \tan^{-1}\left(\frac{x_0}{v_0} \cdot w_n\right)$$

(a), (b), $\Rightarrow X = \sqrt{(2)^2 + \left(\frac{1}{10}\right)^2} \Rightarrow X = 2.0025\ mm$

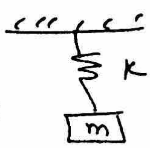
(a) $\phi = \tan^{-1}\left(\frac{2}{1} \cdot 10\right) \Rightarrow \phi = 87.15\ deg$

(b) $\phi = \tan^{-1}\left(\frac{-2}{1}\ 10\right) \Rightarrow \phi = -87.15\ deg.$

Example 3

$m = 49.2\ g,\ k = 857.8\ N/m$

$x_0 = 10\ mm,\ v_0 = 0\ mm/s$



Find

1. Natural Frq (w_n)
2. Natural period (T_n)
3. Maximum → Displacement amplitude / → Velocity " / → Acceleration "
4. ϕ

Solution

1. $w_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} \Rightarrow w_n = 132\ rad/s$

2. $T_n = \frac{1}{f_n} = \frac{2\pi}{w_n} = \frac{2\pi}{132} \Rightarrow T_n = 0.0476\ Sec$

3. Disp $x(t) = X \sin(w_n t + \phi)$

$$X = \sqrt{x_0^2 + \left(\frac{v_0}{w_n}\right)^2} = x_0 \Rightarrow X = 10\ mm$$

Velocity $\Rightarrow \dot{x}(t) = \underbrace{w_n X}_{V \text{ velocity Amp}} \cos(w_n t + \phi)$

$$V = w_n X = (132)(10) = 1320\ mm/s$$

Acceleration $\ddot{x}(t) = \underbrace{-w_n^2 X}_{A \text{ accel amplitude}} \sin(w_n t + \phi)$

$$A = w_n^2 X = (132)^2 (10) = 174.24\ m/s^2$$

4. $\phi = \tan^{-1}\left(\frac{x_0}{v_0} w_n\right) \Rightarrow \phi = \tan^{-1}(\infty)$

$\phi = 90°$

...mple

For pendulum system
① T_n = 3 Seconds , find length (ℓ) , $g = 9.81\ m/s^2$

$$T_n = \frac{2\pi}{w_n} \quad , \quad w_n = \sqrt{\frac{g}{\ell}}$$

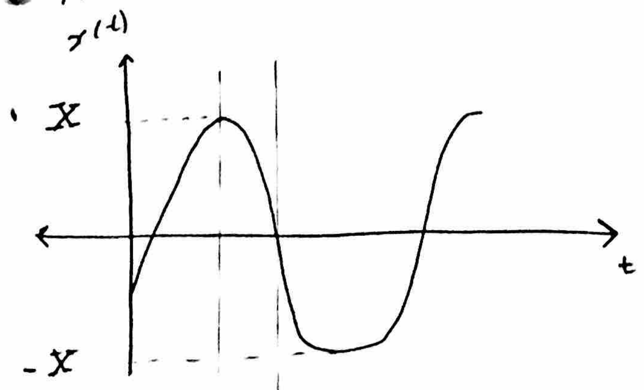
$$\Rightarrow T_n = \frac{2\pi\sqrt{\ell}}{\sqrt{g}} \Rightarrow \ell = \frac{g T_n^2}{(2\pi)^2} = \frac{(9.81)(3)}{(2\pi)^2} \Rightarrow \ell = 2.237\ m$$

- Harmonic Motion

Displacement $\quad x(t) = X \sin(w_n t + \phi)$

Velocity $\quad \dot{x}(t) = w_n X \cos(w_n t + \phi) = V \cos(w_n t + \phi)$

Acceleration $\quad \ddot{x}(t) = -w_n^2 X \sin(w_n t + \phi) = A \sin(w_n t + \phi)$



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$x(t) = 0 \rightarrow$ max $\dot{x}(t)$
$\quad\quad\quad\ \rightarrow \ddot{x}(t) = 0$

$x(t)$ max/min $\rightarrow \dot{x}(t) = 0$
$\quad\quad\quad\quad\quad \rightarrow$ min/max $\ddot{x}(t)$

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