1

* Transverse Vibration of Beams - Ch 11

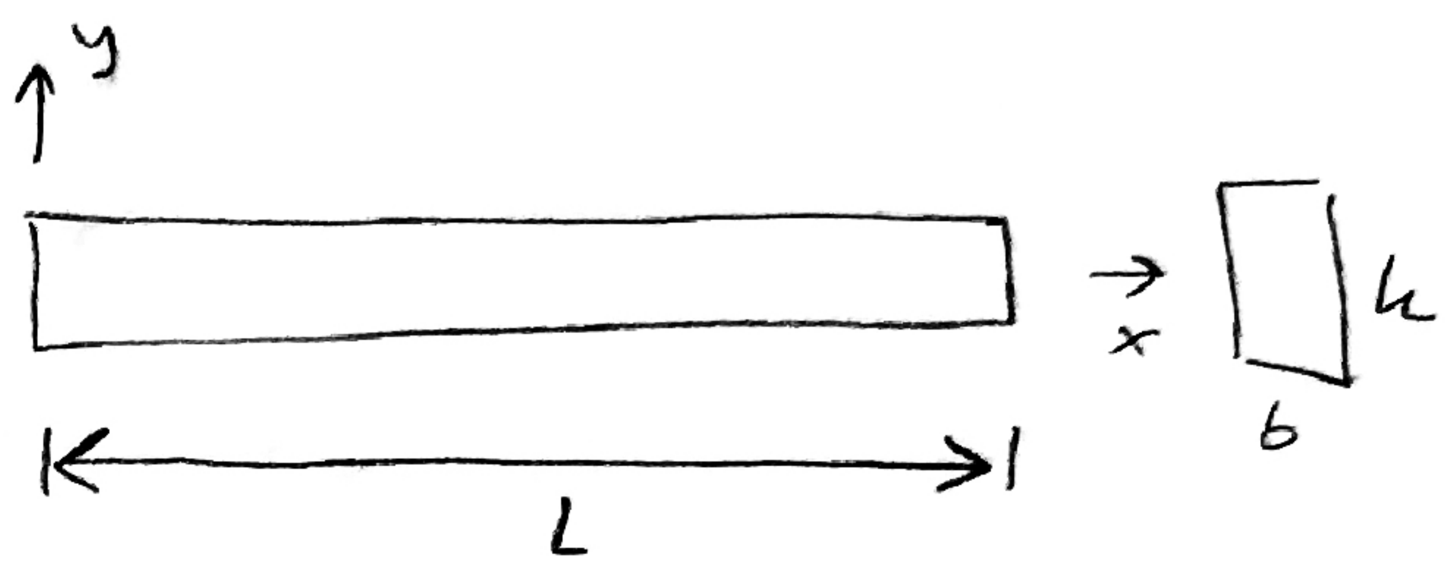
11.1

1- Euler-Bernoulli beam theory (only bending deformation) → we will deal with this.

2- Timoshenko beam theory (Bending and shear deformations)

* Euler-Bernoulli beam theory assumptions

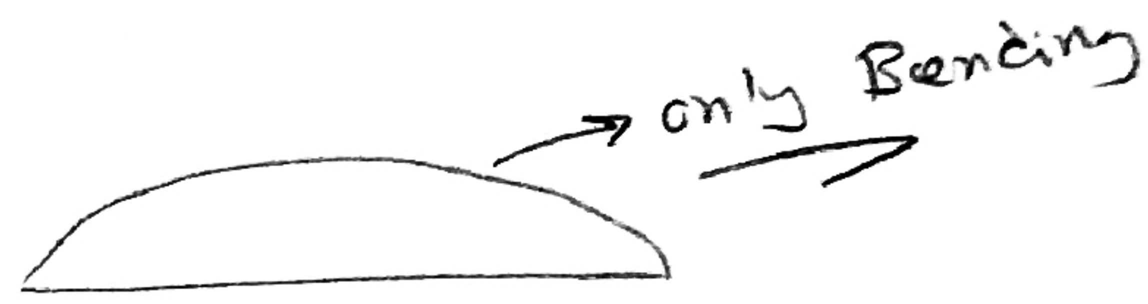
① $L \geq b$ or h by ten times



② Deflections are too small compared to the beam length (L)

L: length, b: width, h: height

③ Deflection function of x and time (t) is $w(x,t)$.



④ Plane section remains plane.

11.2

* Kinetic energy and Potential Energy:

$$T = \frac{1}{2} \int_0^l \rho A \, \dot{w}^2(x,t) \, dx \qquad \dot{w}(x,t) = \frac{\partial w}{\partial t}$$

$$V = \frac{1}{2} \int_0^l E I \, w_{xx}^2(x,t) \, dx \qquad w_{xx}(x,t) = \frac{\partial^2 w}{\partial x^2}$$

ρ: Density (kg/m^3)

E: Elastic Modulus (Pa)

A: Cross-sectional area ($bh = A$) [m^2]

I: Moment of Inertia about centroid [m^4] $= \frac{1}{12} b h^3$

②

Equation of motion (Conservative system — no damping or Forces)

$$\rho A \ddot{w}(x,t) + EI\, w_{xxxx}(x,t) = 0$$

$$\ddot{w} = \frac{\partial^2 w}{\partial t^2}$$

$$w_{xxxx} = \frac{\partial^4 w}{\partial x^4}$$

* Non-conservative system

$$\rho A \ddot{w} + EI\, w_{xxxx} + C \dot{w} = F(t)$$

* Boundary conditions

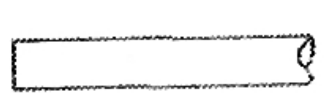
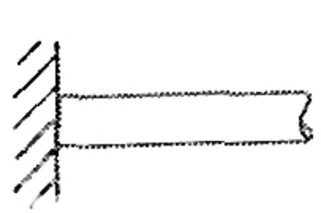
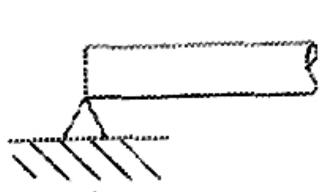
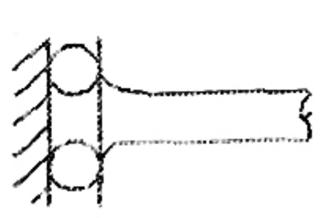
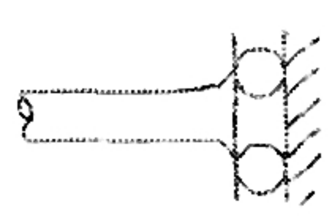
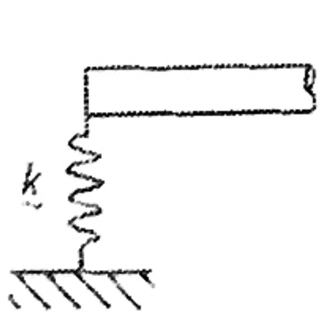
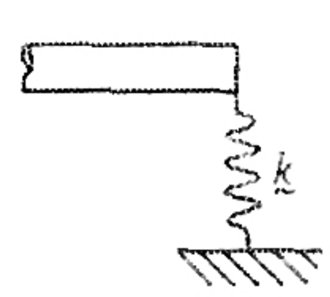
Intial conditions

$w(x,0) = w_0$

$\dot{w}(x,0) = \dot{w}_0$

Table 11.1 — textbook 1 page 323.

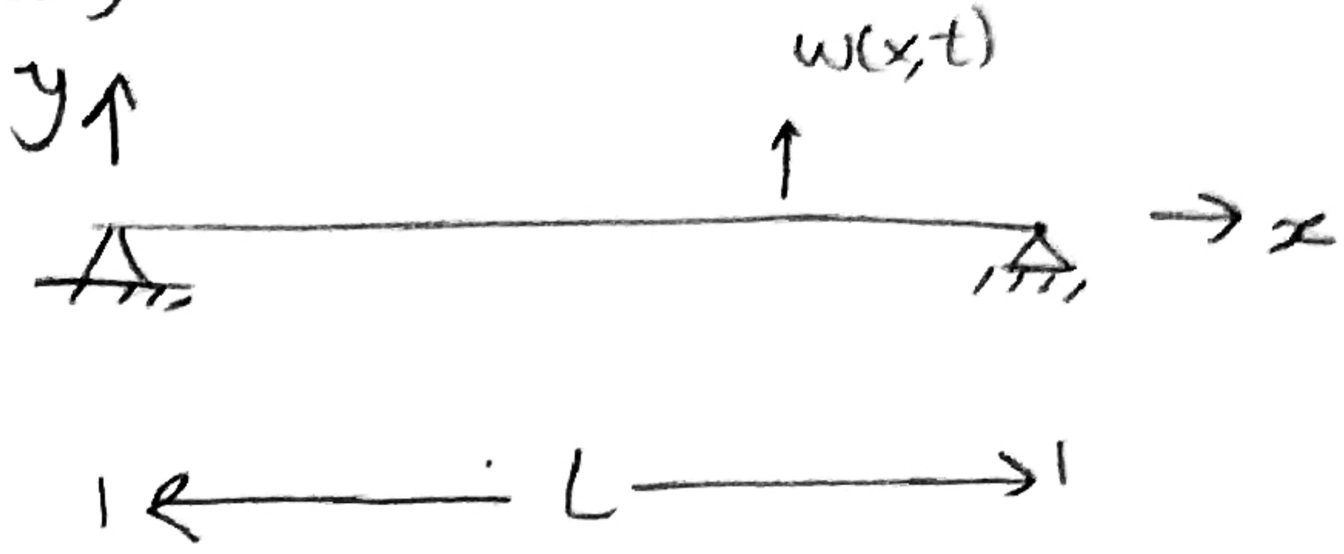
Table 11.1 Boundary Conditions of a Beams[†]

Boundary condition		At left end ($x = 0$)		At right end ($x = l$)
1. Free end (bending moment = 0, shear force = 0)	 $x=0$	$EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)} = 0$	 $x=l$	$EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(l,t)} = 0$
2. Fixed end (deflection = 0, slope = 0)	 $x=0$	$w(0,t) = 0$ $\frac{\partial w}{\partial x}(0,t) = 0$	 $x=l$	$w(l,t) = 0$ $\frac{\partial w}{\partial x}(l,t) = 0$
3. Simply supported end (deflection = 0, bending moment = 0)	 $x=0$	$w(0,t) = 0$ $EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$	 $x=l$	$w(l,t) = 0$ $EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$
4. Sliding end (slope = 0, shear force = 0)	 $x=0$	$\frac{\partial w}{\partial x}(0,t) = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)} = 0$	 $x=l$	$\frac{\partial w}{\partial x}(l,t) = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(l,t)} = 0$
5. End spring (spring constant = k)	 $x=0$	$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert_{(0,t)} = -kw(0,t)$ $EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$	 $x=l$	$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\vert(l,t) = kw(l,t)$ $EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$

(continued overleaf)

* Boundary conditions for common beam types:

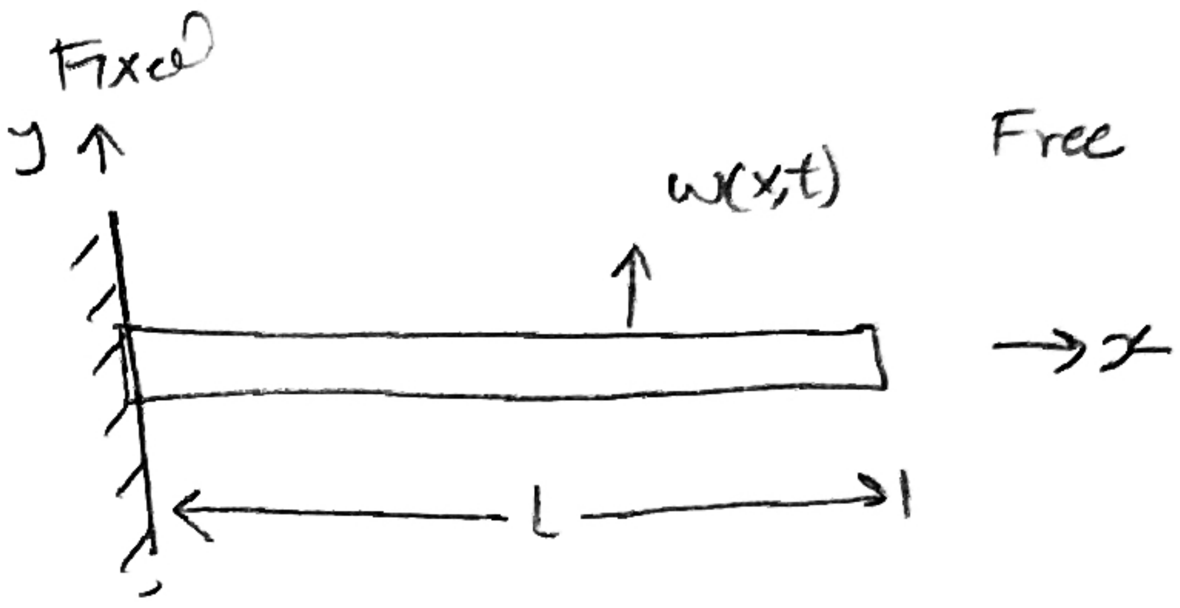
① Simply-Supported Beams (Pinned - Pinned)



$w(0,t) = w(l,t) = 0$

$w_{xx}(0,t) = w_{xx}(l,t) = 0 \qquad w_{xx} = \frac{\partial^2 w}{\partial x^2}$

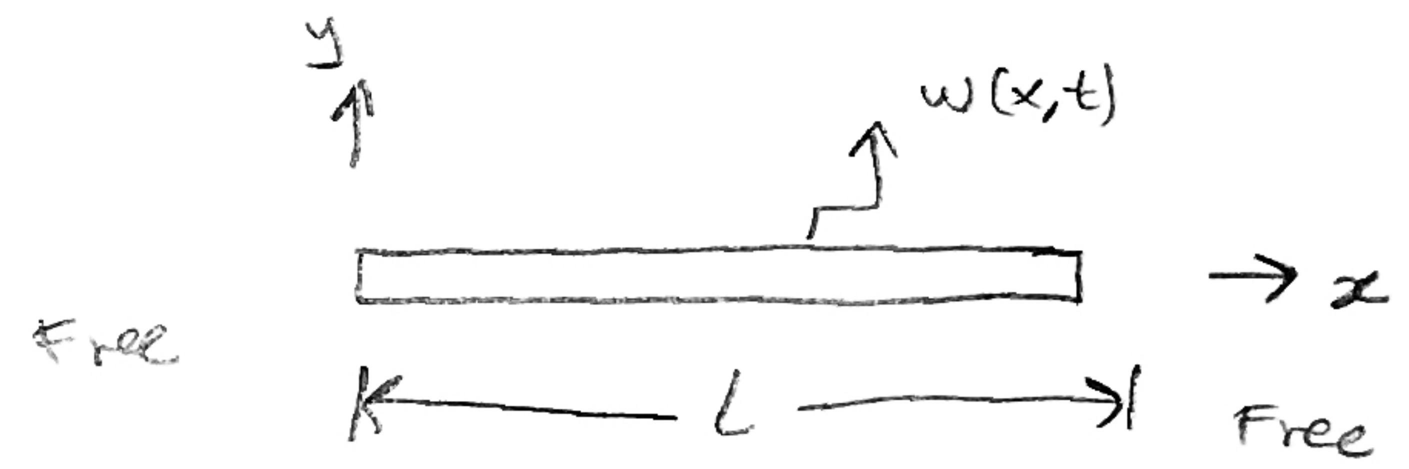
② Cantilever Beam (Fixed - Free)



$w(0,t) = 0 \; , \; w_x(0,t) = 0$ Fixed end

$w_{xx}(l,t) = 0 \; , \; w_{xxx}(l,t) = 0$ Free end

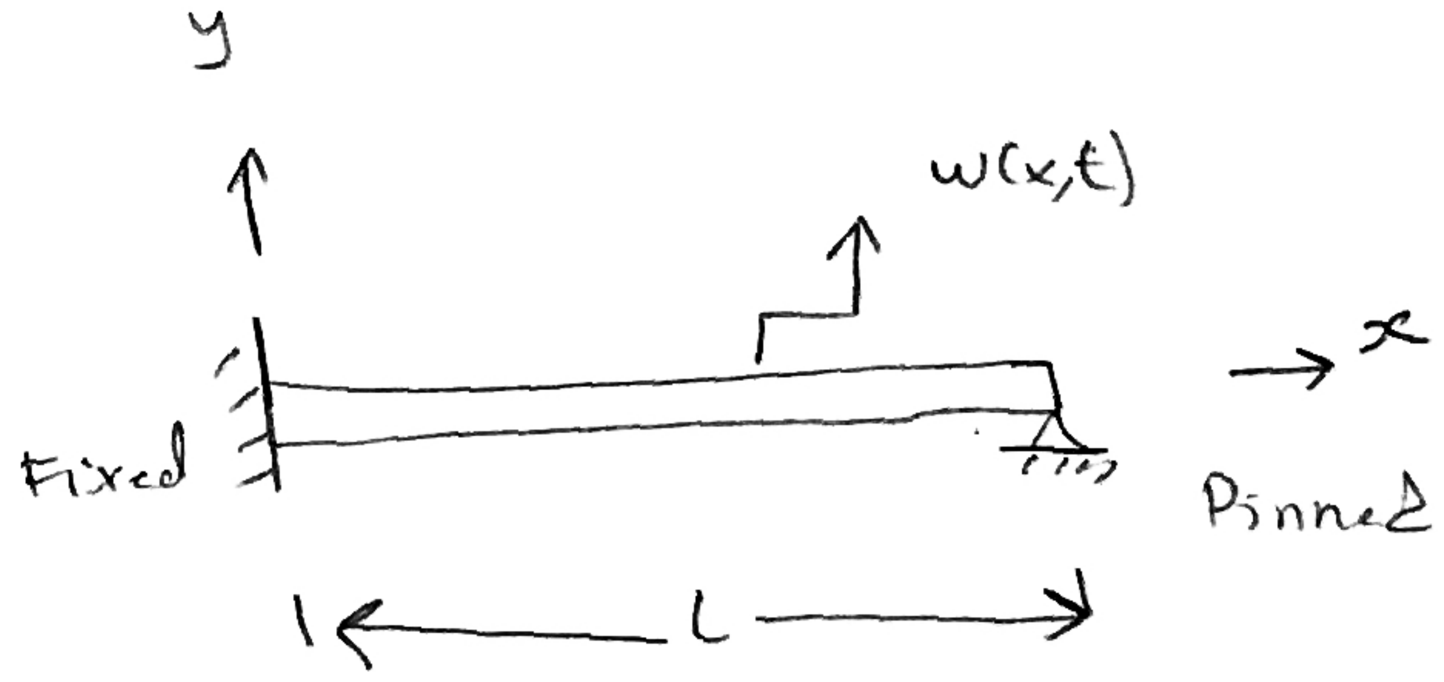
③ Free - Free beam



$w_{xx}(0,t) = w_{xx}(l,t) = 0$

$w_{xxx}(0,t) = w_{xxx}(l,t) = 0$

④ Pinned - Fixed beam



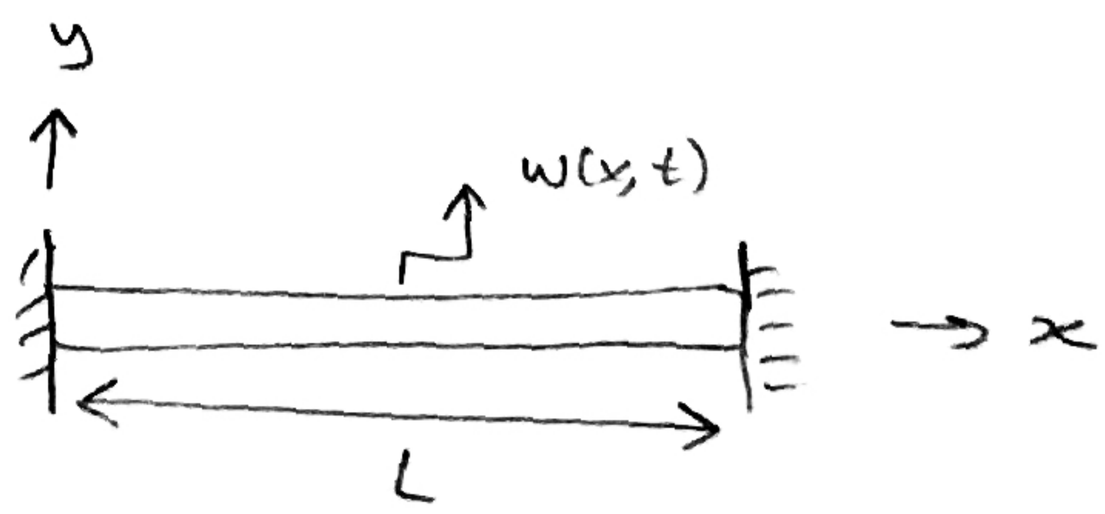
$w(0,t) = w_x(0,t) = 0$ Fixed end

$w(l,t) = w_{xx}(l,t) = 0$ Pinned end

⑤ Fixed - Fixed Beams (Clamped - Clamped)

$w(0,t) = w(l,t) = 0$

$w_x(0,t) = w_x(l,t) = 0$



11.4 Free vibration solution

$$EI\, w_{xxxx} + \rho A \ddot{w} = 0 \qquad -(1)$$ Partial Differential equation (PDE)

seperation of variables

$$w(x,t) = X(x)\,\eta(t) \leftarrow$$ Substitute in Eq(1)

$$EI\, \phi_{xxxx}^{(x)}\, \eta(t) + \rho A\, \phi(x)\, \ddot{\eta}(t) = 0$$

$$\frac{\ddot{\eta}}{\eta} = -\frac{EI}{\rho A} \cdot \frac{\phi_{xxxx}}{\phi} = -\omega^2$$ "omega"

constant → to get harmonic solution

let $C^2 = \frac{EI}{\rho A}$

$$\frac{\ddot{\eta}}{\eta} = -C^2 \frac{\phi_{xxxx}}{\phi} = -\omega^2$$

$$\Rightarrow \ddot{\eta} + \omega^2 \eta = 0 \Rightarrow \eta(t) = A\cos\omega t + B\sin\omega t$$ A, B From IC's

$$\phi_{xxxx}^{(x)} - \left(\frac{\omega}{C}\right)^2 \phi = 0$$ let $p^4 = \left(\frac{\omega}{C}\right)^2 = \omega^2 \frac{\rho A}{EI}$

$$\omega = p^2 \sqrt{\frac{EI}{\rho A}}$$

→ Depends on BC's

$$\phi_{xxxx} - p^4 \phi = 0 \qquad -(2)$$

$\phi(x) = a\, e^{rx}$ in Eq(2)

$$r^4 - p^4 = 0 \Rightarrow r_{1,2} = \pm p$$
$$r_{3,4} = \pm \hat{i} p$$

$$\phi(x) = A e^{px} + B e^{-px} + C e^{\hat{i}px} + D e^{-\hat{i}px}$$

or

$$\phi(x) = a\cos(px) + b\sin(px) + c\cosh(px) + d\sinh(px)$$

a, b, c, d From BC's

Modeshape

$$w(x,t) = \sum_{i=1}^{n} \phi_i(x)\, \eta_i(t)$$

$$\phi_i(x) = a\cos(p_i x) + b\sin(p_i x) + c\cosh(p_i x) + d\sinh(p_i x)$$
$$\eta_i(t) = A_i \cos\omega_i t + B_i \sin\omega_i t$$

Natural Frequencies and mode shapes

11.5.1 Simply-Supported (Pinned-Pinned) Beam

EOM $\quad EI\, w_{xxxx} + \rho A \ddot{w} = 0$

BC's $\quad w(0,t) = w(l,t) = 0$

$\quad w_{xx}(0,t) = w_{xx}(l,t) = 0$

Find Nat. Freq and modeshapes

$w(x,t) = \phi(x)\,\eta(t)$

$\Rightarrow \phi(x) = a\cos(px) + b\sin(px) + c\cosh(px) + d\sinh(px)$

$\eta(t) = A\cos\omega t + B\sin\omega t$

$p^4 = \left(\frac{c}{\omega}\right)^2, \quad c^2 = \frac{\rho A}{EI}$

BC's

$w(0,t) = \phi(0)\eta(t) = 0 \Rightarrow \phi(0) = 0$

$w(l,t) = \phi(l)\eta(t) = 0 \Rightarrow \phi(l) = 0$

$w_{xx}(0,t) = \phi_{xx}(0)\eta(t) = 0 \Rightarrow \phi_{xx}(0) = 0$

$w_{xx}(l,t) = \phi_{xx}(l)\eta(t) = 0 \Rightarrow \phi_{xx}(l) = 0$

$\phi(0) = 0 = a\cos(0) + b\sin(0) + c\cosh(0) + d\sinh(0)$

$a + c = 0 \quad - (*)$

$\phi_x(x) = -pa\sin(px) + pb\cos(px) + pc\sinh(px) + pd\cosh(px)$

$\phi_{xx}(x) = -p^2 a\cos(px) - p^2 b\sin(px) + p^2 c\cosh(px) + p^2 d\sinh(px)$

$\phi_{xx}(0) = -p^2 a + p^2 c = 0 \quad \rightarrow (**)$

From Equation $(*)$ and $(**)$ $\quad a = c = 0$

$\phi(x) = b\sin(px) + d\sinh(px)$

$\phi(l) = 0 = b\sin pl + d\sinh(pl)$

$\phi_{xx}(l) = 0 = -p^2 b\sin(pl) + p^2 d\sin(pl)$

$$\begin{bmatrix} \sin pl & \sinh p \\ & \end{bmatrix} \begin{Bmatrix} b \\ d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Mix Form

$$\begin{bmatrix} \sin pl & \sinh pl \\ -\sin pl & \sinh pl \end{bmatrix} \begin{Bmatrix} b \\ d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

only true if $b = d = 0$ X No Solution

or $\det[\ ] = 0 \Rightarrow \cancel{2}\sin(pl)\sinh(pl) = 0$

$\boxed{\sin(pl) = 0} \Rightarrow pl = n\pi \Rightarrow \boxed{P_n = \frac{n\pi}{l}}$

or

$\sinh(pl) = 0$

↳ only true if $pl = 0$ or $p = 0$ X No Solution $P \neq 0$

Remember $p^4 = \left(\frac{\omega}{C}\right)^2 \Rightarrow \omega = p^2\sqrt{\frac{EI}{\rho A}}$ or $\omega = (pl)^2\sqrt{\frac{EI}{\rho A l^4}}$

$\Rightarrow \boxed{\omega_n = (n\pi)\sqrt{\frac{EI}{\rho A l^4}}}$ Natural frequency for each mode for simply supported Beam

Back to $\phi(x) = \underbrace{b\sin(px)}_{=0} + \underbrace{\cancel{d}^{\,0}\sinh(px)}_{=0}$ or

$\phi(x) = b\sin(px)$

or $\phi_n(x) = b_n\sin\left(\frac{n\pi}{l}x\right)$ $\qquad P_n = \frac{n\pi}{l}$

In general $b_n = 1$ (Normalized modeshape)

$\phi_n(x) = \sin\left(\frac{n\pi}{l}x\right)$ ⟷ Modeshape of simply-supported Beam

Simply-Supported Beam

$$EI\, w_{xxxx} + \rho A \ddot{w} = 0$$

$$w(0,t) = w(l,t) = 0$$

$$w_{xx}(0,t) = w_{xx}(l,t) = 0$$

$$\Rightarrow \quad \omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} \qquad n = 1, 2, \ldots, n \quad \Rightarrow \quad \omega_n = (n\pi)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

$$\phi(x) = \sin(px) \quad \Rightarrow \quad \phi_n(x) = \sin(p_n x) \quad , \quad p_n = \frac{n\pi}{L}$$

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

First mode shape $(n=1)$

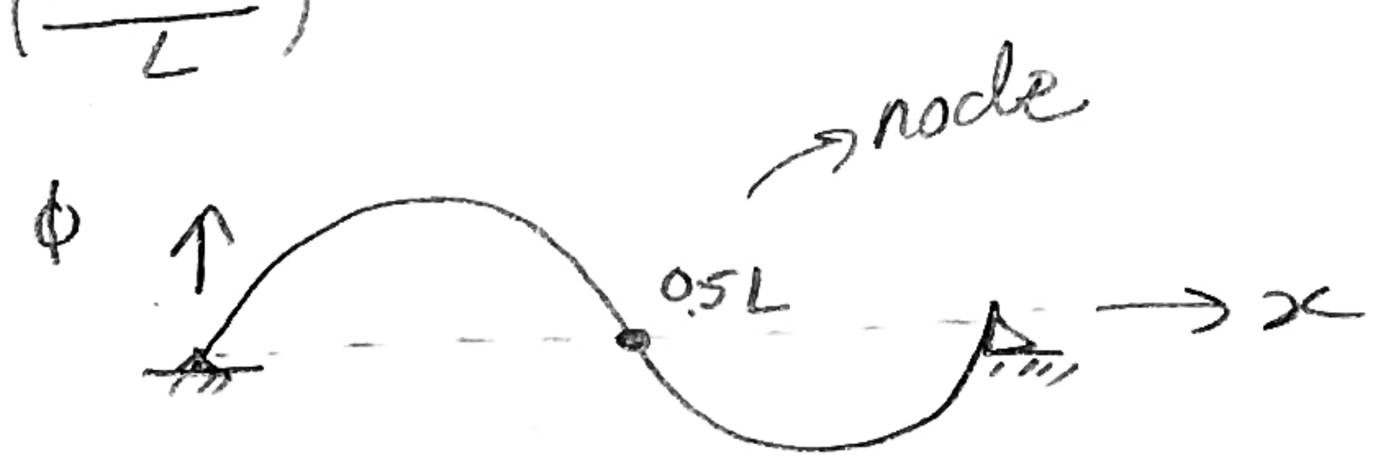
$$\phi_1(x) = \sin\left(\frac{\pi}{L} x\right)$$

$$\omega_1 = (\pi)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

Second Modeshape $n=2$

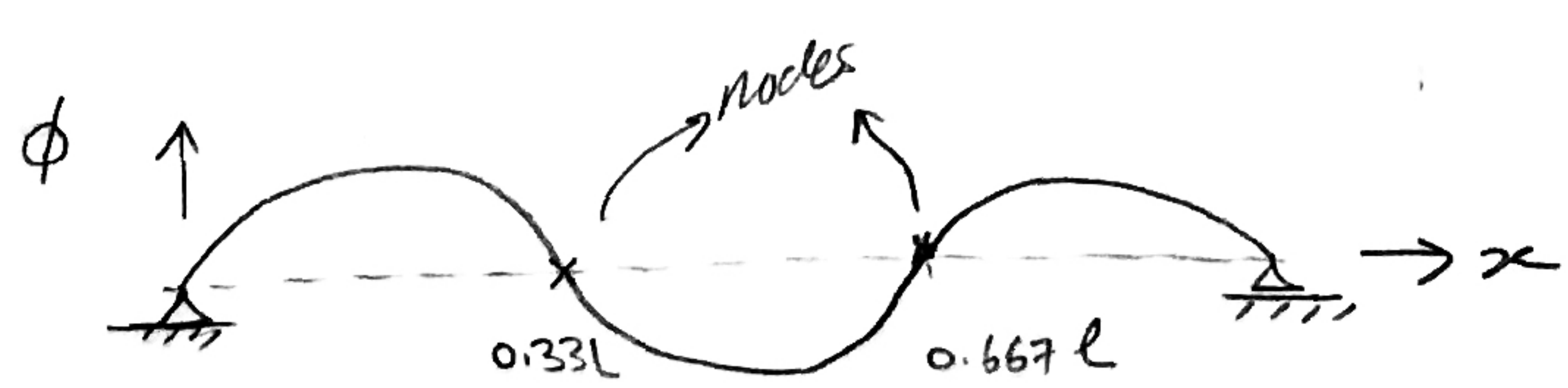
$$\phi_2(x) = \sin\left(\frac{2\pi x}{L}\right)$$

$$\omega_2 = (2\pi)^2 \sqrt{\frac{EI}{\rho A L^4}}$$



third modeshape $(n=3)$

$$\phi_3(x) = \sin\left(\frac{3\pi x}{L}\right)$$



$$\omega_3 = (3\pi)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

Smply-Supported Beam, cont'd

Initial Conditions $w(x,0) = w_0$
$\dot{w}(x,0) = \dot{w}_0$

number of modes

$$w(x,t) = \sum_{i=1}^{n} \phi_n \eta_n(t) \quad , \quad \eta_n(t) = A_n \cos w_n t + B_n \sin w_n t$$

$$\phi_n(x) = \sin w_n t \qquad , \quad w_n = (n\pi)^2 \sqrt{\frac{EI}{\rho A L^4}}$$

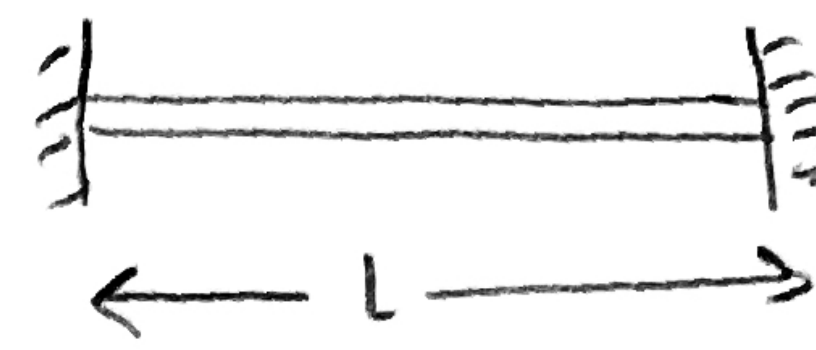
$$= \sum_{i=1}^{n} (\sin w_n x)(A_n \cos w_n t + B_n \sin w_n t)$$

$$w(x,0) = w_0 = \sum_{i=1}^{n} (\sin w_n x) A_n \Rightarrow A_n = \frac{2}{L}\int_0^{l} w_0 \sin\frac{n\pi}{L}x \, dx$$

$$\dot{w}(x,t) = \sum_{i=1}^{n} (w_n)(\sin w_n x)\left[-A_n \sin w_n t + B_n \cos w_n t\right]$$

$$\dot{w}(x,0) = \dot{w}_0 = \sum_{i=1}^{n} w_n \sin w_n x \; B_n \Rightarrow B_n = \frac{2}{w_n L}\int_0^{l} \dot{w}_0 \sin\frac{n\pi x}{L} \, dx$$

Natural Frequencies and mode shapes of Fixed-Fixed Beam



EOM

$$EI\, w_{xxxx} + \rho A \ddot{w} = 0 \qquad -(1)$$

BC's

$$w(0,t) = w(l,t) = 0$$

$$w_x(0,t) = w_x(l,t) = 0$$

Find Nat Freq and mode shapes.

Solution

separation of variables

$$w(x,t) = \phi(x)\eta(t) \qquad \text{sub in eq (1)}$$

$$EI\, \phi_{xxxx}\eta(t) + \rho A \phi \ddot{\eta} = 0$$

$$\frac{\ddot{\eta}}{\eta} = -\overbrace{\frac{EI}{\rho A}}^{c^2} \frac{\phi_{xxxx}}{\phi} = -\omega^2$$

$$\ddot{\eta} + \omega^2 \eta = 0 \Rightarrow \eta(t) = A\cos\omega t + B\sin\omega t$$

$$\phi_{xxxx}(x) - \underbrace{\left(\frac{\omega}{c}\right)^2}_{p^4} \phi(x) = 0 \Rightarrow \phi_{xxxx} - p^4\phi = 0$$

$$\Rightarrow \omega^2 = p^2\sqrt{\frac{EI}{\rho A}}$$

$$\phi(x) = a\cos px + b\sin px + c\cosh px + d\sinh(px)$$

Apply BC's

$$\phi(0) = 0 \Rightarrow \boxed{a + c = 0}$$

$$\phi_x(0) = 0 \Rightarrow \phi_x(x) = -pa\sin px + pb\cos px + pc\sinh px + pd\cosh px$$

$$\phi_x(0) = 0 \Rightarrow pb + pd = 0 \qquad \boxed{b + d = 0}$$

xed- Fixed beam, cont'd

$\phi(l)=0 \Rightarrow a\cos pL + b\sin pL + c\cosh pL + d\sinh pL = 0$

$\phi_x(L)=0 \Rightarrow -pa\sin pL + pb\cos pL + pc\sinh pL + pd\cosh pL = 0$

$-a\sin pL + b\cos pL + c\sinh pL + d\cosh pL = 0$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cos pL & \sin pL & \cosh pL & \sinh pL \\ -\sin pL & \cos pL & \sinh pL & \cosh pL \end{bmatrix}}_{\det[\,]=0} \underbrace{\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}}_{\neq 0} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

If we set $\det[\,]=0 \Rightarrow \cos pL \cosh pL - 1 = 0$

(pL) is root of this equation and could be solved numerically "Newton Raphson method"

$\Rightarrow p_n = \frac{(2n+1)\pi}{2L}$

$$\omega_n = p_n^2 \sqrt{\frac{EI}{\rho A}}$$

$$= \left[\frac{(2n+1)\pi}{2}\right]^2 \sqrt{\frac{EI}{\rho A L^4}}$$

$\Rightarrow$ Assume $a=1 \Rightarrow$ then we can find b, c and d

$$\Rightarrow \phi_n = [\cos p_n x + \cosh p_n x] - \frac{\cos p_n l - \cosh p_n l}{\sin p_n l - \sinh p_n}[\sin p_n x + \sinh p_n x]$$

$$W(x,t) = \sum_{i=1}^{n} \phi_n(x)\,\eta_n(t) \qquad \eta_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t$$