* Lagrange equation of motion and variational approach

For a given system if T is the kinetic energy, V is the potential energy, then the Lagrangian (L) is $L = T - V$. The equation of motion of this system can be expressed as:

Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

where x_i is the degrees of freedom of this system

$i = 1, 2, 3, \ldots, n$ ← # of DOF

* The equation above is used only for conservative systems (no damping, no external forces).

* For non-conservative systems (have damping and/or external forces), we define work (W) which includes both damping and external forces terms. Therefore, the equation of motion can be expressed as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i$$

$Q_i = \frac{\delta W}{\delta x_i}$ → variational parameter works as $\frac{\partial}{\partial x_i}$

$\Rightarrow \delta W = Q_i \, \delta x_i$ → virtual work

In general,

For mass (m)

$$T = \frac{1}{2} m \dot{x}^2$$

multi masses (m_i) $i = 1, 2, \ldots, n$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \cdots + \frac{1}{2} m_n \dot{x}_n^2$$

For spring (k)

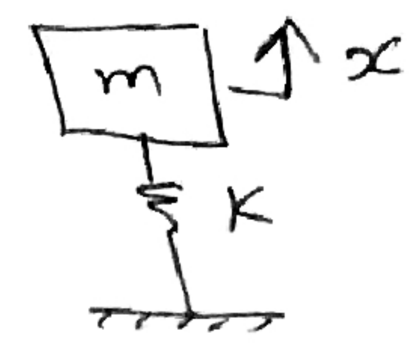
$$V = \frac{1}{2} k x^2$$

multisprings (k_i) $i = 1, 2, \ldots, n$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \cdots + \frac{1}{2} k_n x_n^2$$

Example

Find EOM using Lagrange equation



Solution

$$T = \frac{1}{2} m \dot{x}^2 \quad , \quad V = \frac{1}{2} K x^2$$

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$$

Lagrange Eq.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

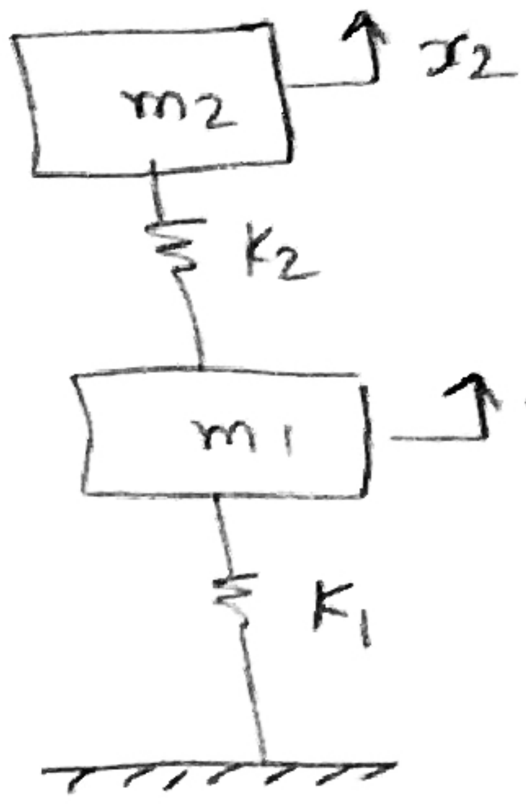
$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2\right) = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

$$\frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2\right) = -kx$$

$$\Rightarrow \quad m\ddot{x} + kx = 0$$

Example: Find EOM using Lagrange Equation



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$\mathcal{L} = T - V =$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2 - x_1)^2$$

Lagrange Eq'n

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \qquad x_i \Rightarrow x_1, x_2 \qquad i = 1,2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \to \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \dot{x}_1$$

$$\to \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \dot{x}_2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) \to \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\to \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial \mathcal{L}}{\partial x_i} \to \frac{\partial \mathcal{L}}{\partial x_1} = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$\to \frac{\partial \mathcal{L}}{\partial x_2} = -k_2 (x_2 - x_1)$$

$i = 1$

$$0 = m_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) \Rightarrow m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

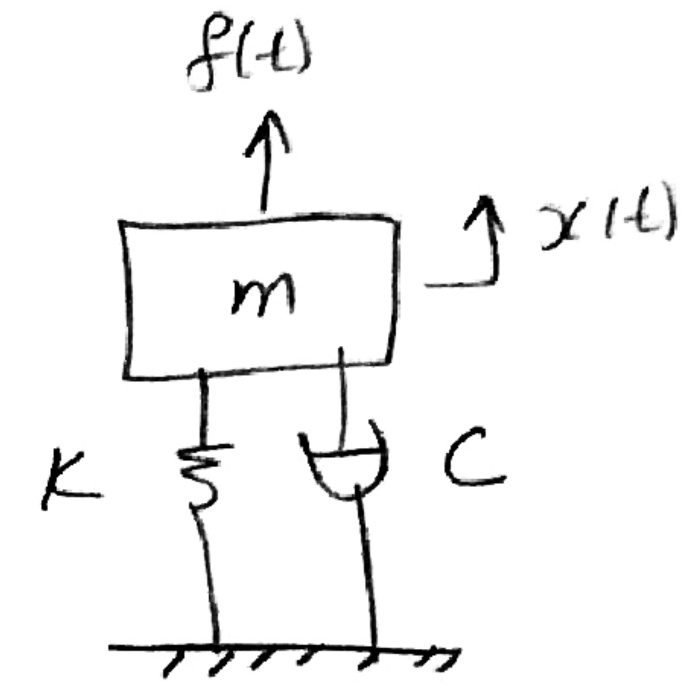
$i = 2$

$$0 = m_2 \ddot{x}_2 + k_2 (x_2 - x_1) \Rightarrow m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

Matrix Form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Example Use Lagrange Eq to find EOM



$T = \frac{1}{2} m \dot{x}^2$, $V = \frac{1}{2} k x^2$

$\delta W = Q_i \, \delta x_i$ $\quad i = 1$

$= Q \, \delta x$ $\quad , \; Q = \underbrace{F_c}_{\text{damping}} + \underbrace{f(t)}_{\text{applied}}$

$= (F_c + f(t)) \, \delta x$ $\quad F_c = -c\dot{x}$

$\delta W = [-c\dot{x} + f(t)] \, \delta x$

$Q = \frac{\delta W}{\delta x} = -c\dot{x} + f(t)$

Lagrange Eq

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x} = Q = -c\dot{x} + f(t)$$

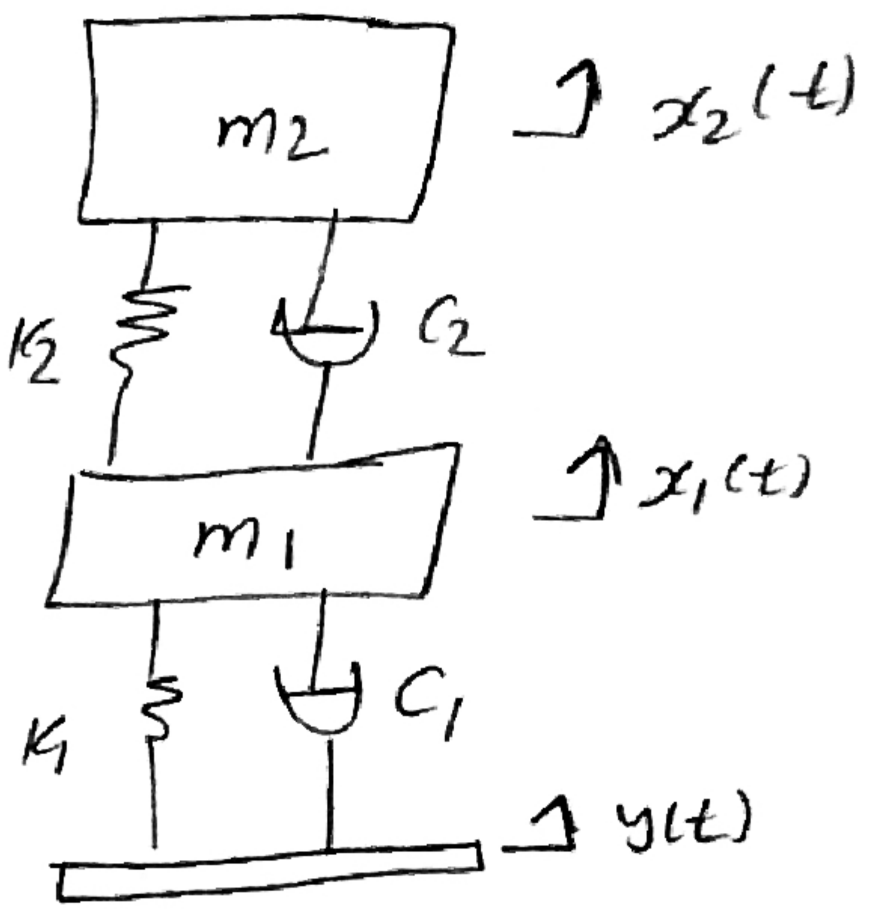
$\frac{dL}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt} = m\ddot{x}$

$\frac{\partial L}{\partial x} = -kx$

$\Rightarrow \quad m\ddot{x} + kx = -c\dot{x} + f(t)$

$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = f(t)}$

Example write Equation of motion using Lagrange equation



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} K_1 (x_1 - y)^2 + \frac{1}{2} K_2 (x_2 - x_1)^2$$

$$\delta W = Q_i \, \delta x_i \quad , \quad i = 1, 2$$

$$= Q_1 \delta x_1 + Q_2 \delta x_2$$

$$= \underbrace{\left[- c_1 (\dot{x}_1 - \dot{y}) + c_2 (\dot{x}_2 - \dot{x}_1)\right]}_{\to Q_1} \delta x_1 + \underbrace{\left[- c_2 (\dot{x}_2 - \dot{x}_1)\right]}_{\to Q_2} \delta x_2$$

$$L = T - V$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} K_1 (x_1 - y)^2 - \frac{1}{2} K_2 (x_2 - x_1)^2$$

Lagrange Eq'n

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i$$

$i = 1$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial L}{\partial x_1} = - K_1 (x_1 - y) + K_2 (x_2 - x_1)$$

$$Q_1 = - c_1 (\dot{x}_1 - \dot{y}) + c_2 (\dot{x}_2 - \dot{x}_1)$$

$$\Rightarrow m_1 \ddot{x}_1 + K_1 (x_1 - y) - K_2 (x_2 - x_1) = - c_1 (\dot{x}_1 - \dot{y}) + c_2 (\dot{x}_2 - \dot{x}_1)$$

$$m_1 \ddot{x}_1 + (K_1 + K_2) x_1 - K_2 x_2 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_2 = K_1 y + c_1 \dot{y}$$

$i = 2$

$$\frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial L}{\partial x_2} = K_2 (x_2 - x_1) \Rightarrow m_2 \ddot{x} + K_2 x_2 - K_2 x_1 = - c_2 (\dot{x}_2 - \dot{x}_1)$$

$$Q_2 = - c_2 (\dot{x}_2 - \dot{x}_1) \Rightarrow m_2 \ddot{x}_2 - K_2 x_1 + K_2 (x_2) + c_2 \dot{x}_2 - c_2 \dot{x}_1 = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

$$F_1 = K_1 y + c_1 \dot{y}$$