Example

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F(t)\}$$

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad C = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix}, \quad [k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}$$

$m_1 = 10\,kg$
$m_2 = 5\,kg$
$m_3 = 1\,kg$

$\zeta_1 = 0.1$
$\zeta_2 = 0.2$
$\zeta_3 = 0.1$

$k_1 = 500\ N/m$
$k_2 = 300\ N/m$
$k_3 = k_4 = 2500\ N/m$

$$\{x(0)\} = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} m \quad (1)$$

$$\dot{x}(0) = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} m/s$$

$$\{F(t)\} = \begin{Bmatrix} F_1(t) \\ 0 \\ 0 \end{Bmatrix}$$

$F_1(t) = \cos 3t$

① Find Nat. Frequencies and Mode shapes

② Responses $x_1(x)$, $x_2(t)$, $x_3(t)$.

① For natural Frequencies and modeshapes ⟹ Drop $[C]$ and $\{F(t)\}$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

$$\{x(t)\} = \{X\}\sin(\omega t + \phi)$$

$$\Rightarrow \left([k] - \omega^2[m]\right)\{X\} = 0 \Rightarrow \left| \begin{bmatrix} 800 & -300 & 0 \\ -300 & 550 & -250 \\ 0 & -250 & 500 \end{bmatrix} - \omega^2 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

Use MATLAB

$[V, D] = eig(k, m)$

$\omega_1^2 = 39$ $\quad \omega_2^2 = 121$ $\quad \omega_3^2 = 530$

$\omega_1 = 6.25$ rad/s $\quad \omega_2 = 11$ rad/s $\quad \omega_3 = 23$ rad/s

$$\{X^{(1)}\} = \begin{Bmatrix} -0.2242 \\ -0.3065 \\ -0.1662 \end{Bmatrix}, \quad \{X^{(2)}\} = \begin{Bmatrix} -0.2229 \\ 0.3043 \\ 0.2007 \end{Bmatrix}, \quad \{X^{(3)}\} = \begin{Bmatrix} 0.0077 \\ -0.1160 \\ 0.9654 \end{Bmatrix}$$

$$[U_n] = \left[\{X^{(1)}\} \quad \{X^{(2)}\} \quad \{X^{(3)}\}\right]$$

$$[U_n] = \begin{bmatrix} -0.2242 & -0.2229 & 0.0077 \\ -0.3065 & 0.3043 & -0.1160 \\ -0.1662 & 0.2007 & 0.9654 \end{bmatrix}$$

$$\{x(t)\} = [U_n]\ \eta_i(t)$$

we know

$$\eta_i(t) = \int_0^t Q_i(\tau)\, h(t-\tau)\, d\tau + g(t)\,\eta_i(0) + h(t)\,\dot{\eta}_i(0)$$

$$Q_i(t) = [U_n]^T \{F(t)\} = \begin{bmatrix} -0.2242 & -0.3065 & -0.1662 \\ -0.2279 & 0.3043 & 0.2007 \\ 0.0077 & -0.1160 & 0.9654 \end{bmatrix} \begin{Bmatrix} F_1(t) \\ 0 \\ 0 \end{Bmatrix}$$

$$Q_i(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix} = \begin{Bmatrix} -0.2242 \\ -0.2279 \\ 0.0077 \end{Bmatrix} F_1(t) = \cos 3t$$

$$\eta_i(0) = [U_n]^T [m] \{x(0)\} = \begin{Bmatrix} -1.6986 \\ 1.7223 \\ 0.3853 \end{Bmatrix}$$

$$\dot{\eta}_i(0) = [U_n]^T [m] \{\dot{x}(0)\} = \begin{Bmatrix} -2.2421 \\ -2.2287 \\ 0.0773 \end{Bmatrix}$$

$$g(t) = e^{-\zeta_i \omega_i t}\left(\cos \omega_{di} t + \frac{\zeta_i \omega_i}{\omega_{di}} \sin \omega_{di} t\right)$$

$$h(t) = \frac{1}{\omega_{di}} e^{-\zeta_i \omega_i t} \sin \omega_{di} t$$

$$\omega_{di} = \omega_i \sqrt{1-\zeta_i^2}$$

$$\omega_{d1} = 6.2830 \text{ rad/s}$$

$$\omega_{d2} = 10.9432$$

$$\omega_{d3} = 22.9073$$

$$\Rightarrow \eta_1(t) = \int_0^t (-0.2242 \cos 3\tau)\, h(t-\tau)\, d\tau - 1.6986\, g(t) - 2.2421\, h(t)$$

$$\eta_2(t) = \int_0^t (-0.2279 \cos 3\tau)\, h(t-\tau)\, d\tau + 1.7223\, g(t) - 2.2287\, h(t)$$

$$\eta_3(t) = \int_0^t (0.0773 \cos 3\tau)\, h(t-\tau)\, d\tau + 0.3853\, g(t) + 0.0773\, h(t)$$

$$\Rightarrow \{x(t)\} = [U_n]\{\eta(t)\} \Rightarrow \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} = \begin{bmatrix} -0.2242 & -0.2279 & 0.0077 \\ -0.3065 & 0.3043 & -0.1160 \\ -0.1662 & 0.2007 & 0.9654 \end{bmatrix} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \\ \eta_3(t) \end{Bmatrix}$$

$$x_1(t) = -0.2242\,\eta_1(t) - 0.2279\,\eta_2(t) + 0.0077\,\eta_3(t)$$

$$x_2(t) = -0.3065\,\eta_1(t) + 0.3043\,\eta_2(t) - 0.1160\,\eta_3(t)$$

$$x_3(t) = -0.1662\,\eta_1(t) + 0.2007\,\eta_2(t) + 0.9654\,\eta_3(t)$$

```
% 3 DOF Example
% Special Topics in ME
% Dr. Mohammad A Gharaibeh 20-Feb-2019
clc
clear all
close all
m1 = 10; %kg
m2 = 5; %kg
m3 = 1; %kg

k1 = 500; %N/m
k2 = 300; %N/m
k3 = 250; %N/m
k4 = 250; %N/m

zeta1 = 0.1;
zeta2 = 0.2;
zeta3 = 0.1;

x0 = [0;1;1];
xdot0 = [1;0;0];

% Mass Matrix
M = [m1 0 0;0 m2 0; 0 0 m3]

% Stiffness Matrix
K = [k1+k2 -k2 0;-k2 k2+k3 -k3;0 -k3 k3+k4]

% Find Natural Frequencies and Mode Shapes
[V,D]=eig(K,M)

% Check Normalization
Mcheck = V.'*M*V
Kcheck = V.'*K*V

% Initial Conditions
eta_i_0 = V.'*M*x0
etadot_i_0 = V.'*M*xdot0

% Damped Natural Frequency [rad/sec]
wd1 = sqrt(D(1,1))*sqrt(1-zeta1^2)
wd2 = sqrt(D(2,2))*sqrt(1-zeta1^2)
wd3 = sqrt(D(3,3))*sqrt(1-zeta1^2)
```

$$[C] = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix}$$

$$[C] = \alpha[m] + \beta[k] \quad (1)$$

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad [K] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}$$

$m_1 = 10\ kg$ , $m_2 = 5\ kg$, $m_3 = 1\ kg$

$k_1 = 500\ N/m$ , $k_2 = 300\ N/m$ , $k_3 = k_4 = 250\ N/m$

Find $[C]$, If $f_1 = 0.1$ , $w_1 = 10\ rad/s$

$f_2 = 0.2$ , $w_2 = 30\ rad/s$

Solution

$$2 f_i w_i = \alpha + \beta w_i$$

① $2 f_1 w_1 = \alpha + \beta w_1$

$(2)(0.1)(10) = \alpha + 10\beta \Rightarrow \quad 2 = \alpha + 10\beta$

② $2 f_2 w_2 = \alpha + \beta w_2$

$(2)(0.2)(30) = \alpha + 30\beta \Rightarrow \quad 12 = \alpha + 30\beta$

$$\begin{bmatrix} 1 & 10 \\ 1 & 30 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{bmatrix} 1 & 10 \\ 1 & 30 \end{bmatrix}^{-1} \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} -3 \\ 0.5 \end{Bmatrix}$$

$\alpha = -3$ , $\beta = 0.5$

$$[C] = \alpha[m] + \beta[k] = \begin{bmatrix} 370 & -150 & 0 \\ -150 & 260 & -125 \\ 0 & -125 & 247 \end{bmatrix}$$

$$= \begin{bmatrix} C_1+C_2 & -C_2 & 0 \\ -C_2 & C_2+C_3 & -C_3 \\ 0 & -C_3 & C_3+C_4 \end{bmatrix}$$

$\Rightarrow$ $C_2 = 150$ N-S/m

$C_3 = 125$ N-S/m

$C_1 + C_2 = 370 \Rightarrow C_1 = 370 - 150 \Rightarrow C_1 = 220$ N-S/m

$C_3 + C_4 = 247 \Rightarrow C_4 = 247 - 125 \Rightarrow C_4 = 122$ N-S/m