

2.2.4 Forced Vibration undamped MDOF systems

①

EOM

$$[m]\{\ddot{x}\} + [k]\{x\} = \{f(t)\}$$

Normalized modal matrix
 $\{x(t)\} = [U_n] \eta_i(t)$
 $\eta_i(t) = A_i \cos \omega_i t + B_i \sin \omega_i t$

substitute in EOM and premultiply by $[U_n]^T$

$$\underbrace{[U_n]^T [m] [U_n]}_{[I]} \ddot{\eta}_i + \underbrace{[U_n]^T [k] [U_n]}_{[\omega_i^2]} \eta_i = \underbrace{[U_n]^T \{f(t)\}}_{Q_i(t) = \{Q(t)\} = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{Bmatrix}}$$

$$\ddot{\eta}_i + \omega_i^2 \eta_i = Q_i(t)$$

we can use the convolution Integral to find $\eta_i(t)$

$$\eta_i(t) = \int_0^t Q_i(\tau) h(t-\tau) d\tau + g(t) \eta_i(0) + h(t) \dot{\eta}_i(0)$$

Remember.

$$\eta_i(0) = [U_n]^T [m] \{x(0)\} \rightarrow \text{Initial conditions.}$$

$$\dot{\eta}_i(t) = [U_n]^T [m] \{\dot{x}(0)\}$$

$$g(t) = \cos \omega_i t$$

$$h(t) = \frac{1}{\omega_i} \sin \omega_i t$$

2.2.5 Forced vibration of damped MDOF systems (Proportional Damping) ②

EOM

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{f(t)\}$$

For proportional damping

$$[c] = \alpha [m] + \beta [k]$$

function of mass and stiffness matrices.

$$\Rightarrow [m]\{\ddot{x}\} + (\alpha [m] + \beta [k])\{\dot{x}\} + [k]\{x\} = \{f(t)\} \quad (*)$$

Solution

$$\{x(t)\} = [U_n] \eta_i(t)$$

$[U_n] \rightarrow$ Normalized modal matrix

$$\eta_i(t) = A_i \cos \omega_i t + B_i \sin \omega_i t$$

Substitute in Eq(*) and Premultiply by $[U_n]^T$.

$$\Rightarrow \underbrace{[U_n]^T [m] [U_n]}_I \ddot{\eta}_i + \left(\underbrace{\alpha [U_n]^T [m] [U_n]}_I + \underbrace{\beta [U_n]^T [k] [U_n]}_{[\omega_i^2]} \right) \dot{\eta}_i + \underbrace{[U_n]^T [k] [U_n]}_{[\omega_i^2]} \eta_i = \underbrace{[U_n]^T \{f\}}_{Q_i}$$

$$\ddot{\eta}_i + (\alpha [I] + \beta [\omega_i^2]) \dot{\eta}_i + [\omega_i^2] \eta_i = Q_i(t)$$

$$\text{let } 2\zeta_i \omega_i = \alpha + \beta \omega_i \quad i=1, 2, \dots, n$$

$$\ddot{\eta}_i + 2\zeta_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = Q_i(t)$$

$$\eta_i(t) = \int_0^t Q_i(\tau) h(t-\tau) d\tau + g(t) \eta_i(0) + h(t) \dot{\eta}_i(0)$$

$$g(t) = e^{-\zeta_i \omega_i t} \left(\cos \omega_{di} t + \frac{\zeta_i \omega_i}{\omega_{di}} \sin \omega_{di} t \right), \quad h(t) = \frac{1}{\omega_{di}} e^{-\zeta_i \omega_i t} \sin \omega_{di} t$$

$$\omega_{di} = \omega_i \sqrt{1 - \zeta_i^2}$$

Remember

$$\eta_i(0) = [U_n]^T [m] \{x(0)\}$$

$$\dot{\eta}_i(0) = [U_n]^T [m] \{\dot{x}(0)\}$$

Example

$$m_1 = 200 \text{ kg}$$

$$m_2 = 250 \text{ kg}$$

$$K_1 = 150 \text{ MN/m}$$

$$K_2 = 75 \text{ MN/m}$$

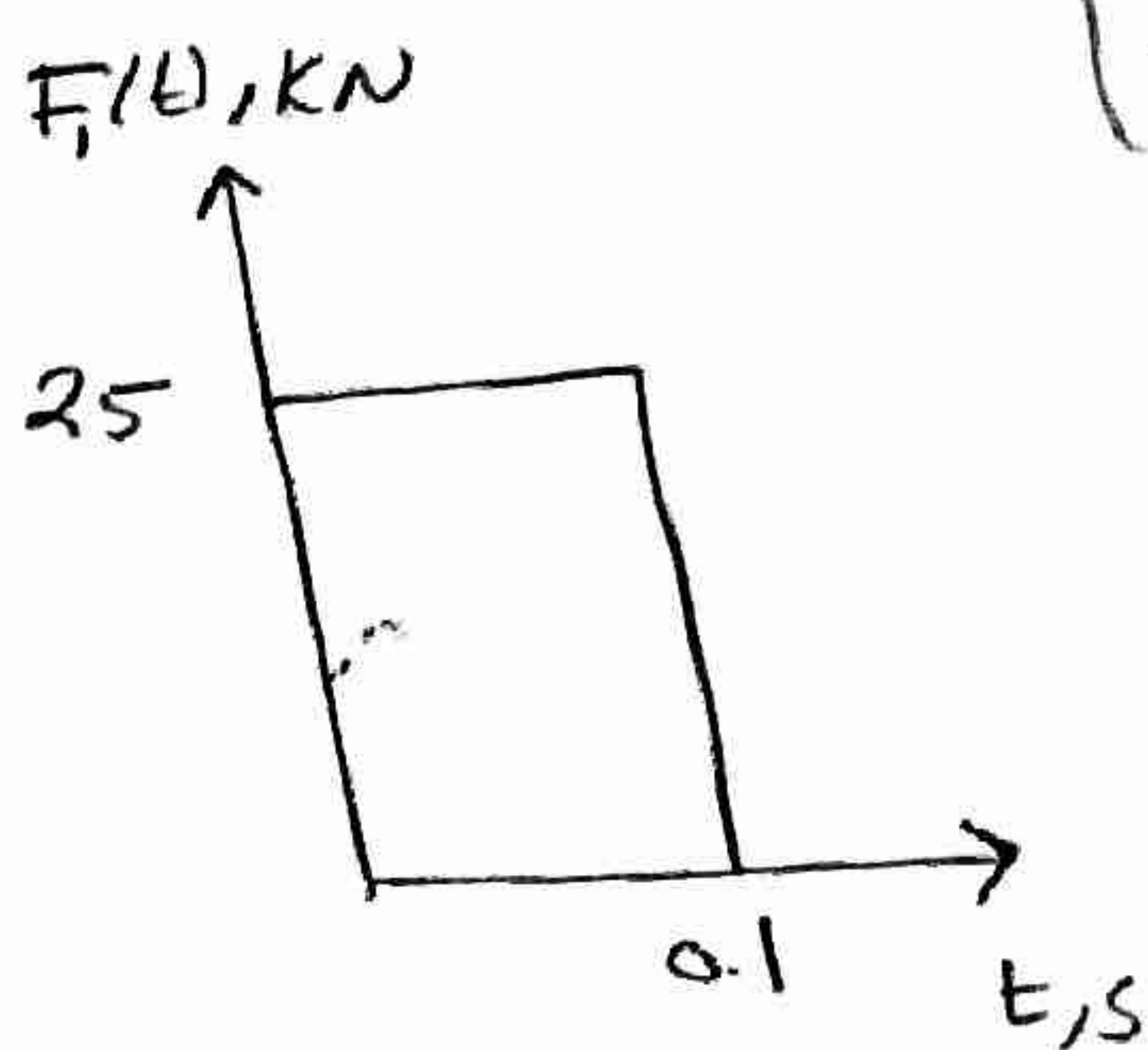
$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

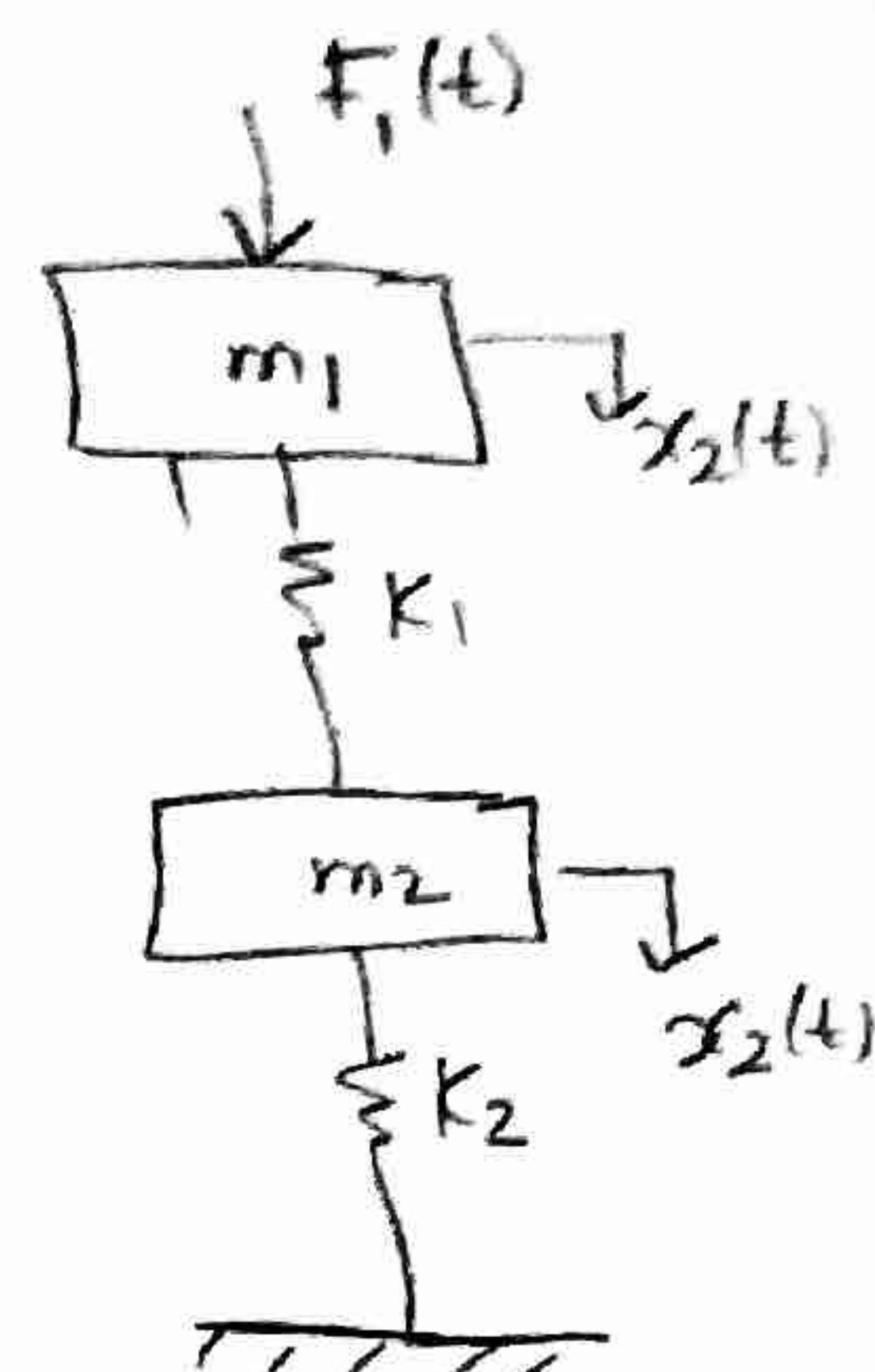
$$[m] \{\ddot{x}\} + [k] \{x\} = \{F_1(t)\}$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[k] = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 + K_2 \end{bmatrix}$$



$$F_1(t) = \begin{cases} 25 \text{ kN} & 0 \leq t \leq 0.1 \text{ sec} \\ 0 & t > 0.1 \text{ sec} \end{cases}$$



Find ① Natural Frequencies and modeshapes

② Express responses (x_1 and x_2) in terms of convolution Integral

Solution

$$\{x\} = \{X\} \sin(\omega t + \phi)$$

$$\Rightarrow [k] - \omega^2 [m] \{X\} = 0$$

det = 0

For natural frequency and mode shape calculations, we always drop damping and forces. Just keep $[m]$ and $[k]$

$$\begin{vmatrix} 150 & -150 \\ -150 & 225 \end{vmatrix} - \omega^2 \begin{vmatrix} 200 & 0 \\ 0 & 250 \end{vmatrix} \times 10^6 = 0$$

$$\begin{vmatrix} 150 - \omega^2 \cdot 0.2 & -150 \\ -150 & 225 - \omega^2 \cdot 0.25 \end{vmatrix} = (150 - \omega^2 \cdot 0.2)(225 - \omega^2 \cdot 0.25) - (-150)(-150) = 0$$

$$\Rightarrow \omega_1^2 = 150 (\text{rad/s})^2 \quad \omega_2^2 = (1500) (\text{rad/sec})^2$$

$$\omega_1 = 12.2474 \text{ rad/s} \quad , \quad \omega_2 = 38.7298 \text{ rad/sec}$$

you can use MATLAB $[V, D] = \text{eig}(K, m)$

Mode Shapes ω_1, ω_2

Cont'd

To find modeshapes $\{X\}$

$$\omega_1^2 = 150$$

$$[K] - \omega_1^2 [m] \{X^{(1)}\} = 0 \Rightarrow \begin{bmatrix} 120 & -150 \\ -150 & 187.5 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = 0$$

$$120 X_1^{(1)} - 150 X_2^{(1)} = 0$$

$$X_1^{(1)} = \frac{150}{120} X_2^{(1)} \Rightarrow X_1^{(1)} = 1.25 X_2^{(1)}$$

$$-150 X_1^{(1)} + 187.5 X_2^{(1)} = 0$$

$$X_1^{(1)} = \frac{187.5}{150} X_2^{(1)} \Rightarrow X_1^{(1)} = 1.25 X_2^{(1)}$$

Same Equation $\Rightarrow X_2^{(1)} = 1$

$$\{X^{(1)}\} = \begin{Bmatrix} 1.25 \\ 1 \end{Bmatrix}$$

$$\omega_2^2 = 1500$$

$$\begin{bmatrix} -150 & -150 \\ -150 & -150 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow X_1^{(2)} = -X_2^{(2)} \Rightarrow \text{let } X_2^{(2)} = 1 \Rightarrow \{X^{(2)}\} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\Rightarrow [U] = \begin{bmatrix} \{X^{(1)}\} & \{X^{(2)}\} \end{bmatrix} = \begin{bmatrix} 1.25 & -1 \\ 1 & 1 \end{bmatrix}$$

To normalize $[U] \Rightarrow [U_n]$

$$[U]^T [m] [U] = \begin{bmatrix} 0.5625 & 0 \\ 0 & 0.45 \end{bmatrix} \Rightarrow [U_n] = \begin{bmatrix} \frac{1.25}{\sqrt{a}} & \frac{-1}{\sqrt{b}} \\ \frac{1}{\sqrt{a}} & \frac{1}{\sqrt{b}} \end{bmatrix}$$

$$[U_n] = \begin{bmatrix} 1.667 & -1.4907 \\ 1.333 & 1.4907 \end{bmatrix}$$

Cont'd

$$\eta_i(t) = \int_0^t Q_i(\tau) h(t-\tau) d\tau + g(t) \cancel{\eta_i(0)} + h(t) \cancel{\dot{\eta}_i(0)}$$

$$[U_n]^T [m] [U_n] \ddot{\eta}_i(t) + [U_n]^T [k] [U_n] \eta_i(t) = [U_n]^T \{F(t)\} \quad Q_i(t)$$

$$\underline{0 \leq t \leq 0.1}$$

$$\eta_i(t) = \int_0^t Q_i(\tau) h(t-\tau) d\tau \quad , i=1,2$$

$$\underline{t > 0.1}$$

$$\eta_i(t) = \int_0^{0.1} Q_i(\tau) h(t-\tau) d\tau + \int_{0.1}^t Q_i(\tau) h(t-\tau) d\tau \quad , i=1,2$$

$$Q_i = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix} = \begin{bmatrix} 1.667 & -1.4907 \\ 1.333 & +1.4907 \end{bmatrix} \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix}$$

$$Q_1(t) = 1.667 F_1(t)$$

$$Q_2(t) = 1.333 F_1(t)$$

$$\{x(t)\} = [U_n]^T [m] \eta_i(t)$$

Response for each mass