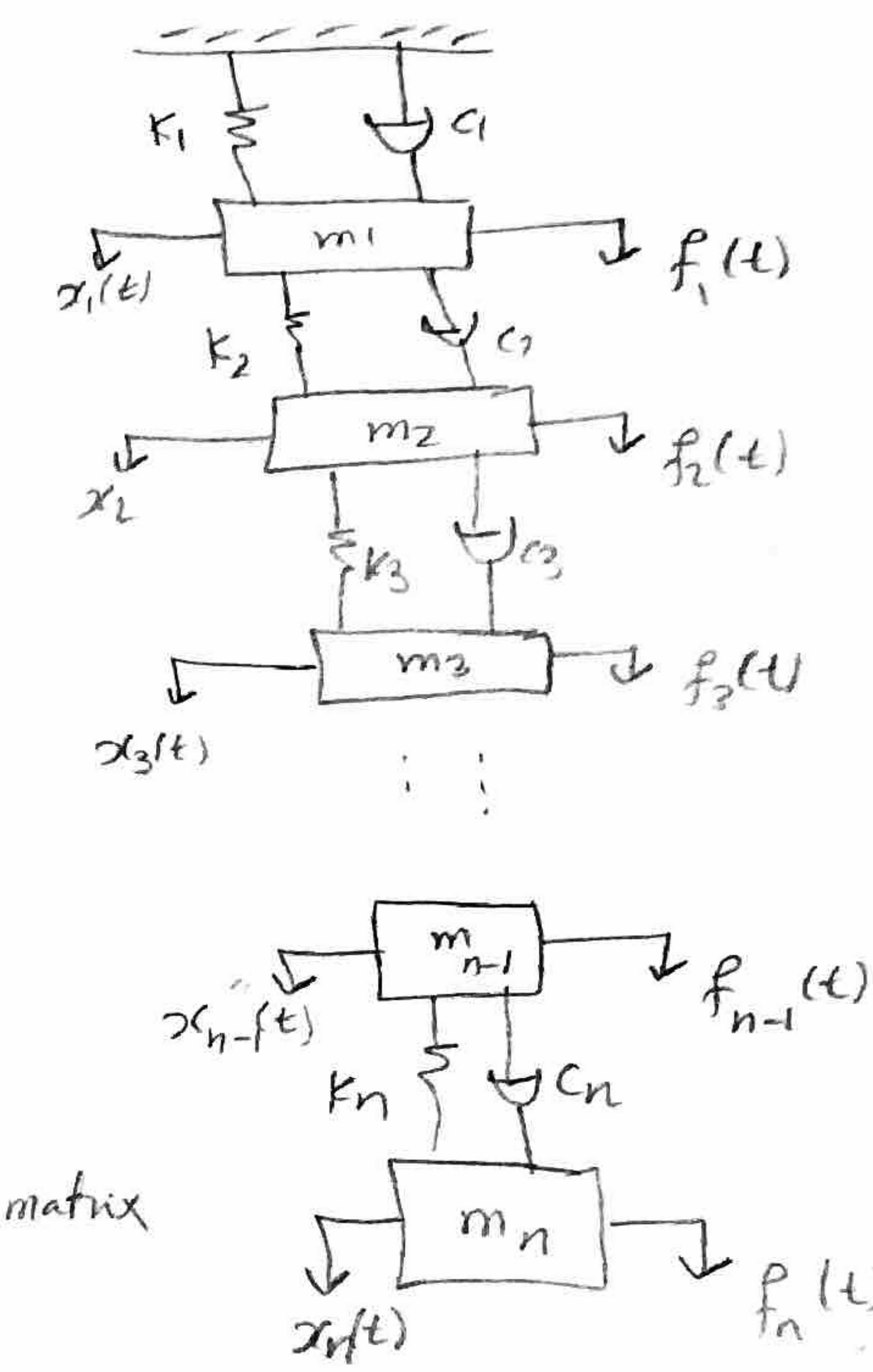
2.2 Vibration of Multidegree-of-freedom systems (MDOF)

EOM

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{f(t)\}$$

where $[m]$, $[c]$ and $[k]$ are mass, damping and stiffness matrices, respectively.

n-degrees of freedom system



$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_n \end{bmatrix}$$ n x n matrix

$$[c] = \begin{bmatrix} c_1+c_2 & -c_2 & 0 & \cdots & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & \cdots & 0 & 0 \\ 0 & -c_3 & c_3+c_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & -c_{n-1} & c_n \end{bmatrix}$$ n x n matrix

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & -k_{n-1} & k_n \end{bmatrix}$$

$\{x\}$, $\{\dot{x}\}$, $\{\ddot{x}\}$ are Displacement, velocity and acceleration vectors

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}_{n\times 1}, \quad \{\dot{x}\} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{Bmatrix}_{n\times 1}, \quad \{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{Bmatrix}_{n\times 1}$$

$\{f(t)\}$ is the Forcing vector

$$\{f(t)\} = \begin{Bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{Bmatrix}_{n\times 1}$$

2.2.1 Eigenvalue Problem

Undamped (MDOF) system (Free vibration)

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

let

$$\{x(t)\} = \{X\} \sin(wt + \phi)$$

$$\{\ddot{x}\} = -w^2\{X\} \sin(wt + \phi)$$

$$\left[[k] - \omega^2[m]\right]\{X\} = 0$$

- For non-trivial solution $(\{X\} \neq 0)$, the determinant of $\left[[k]-\omega^2[m]\right]$ should Equal to zero, So:

$$\left|[k] - \omega^2[m]\right| = 0 \leftarrow \text{Frequency Equation}$$

← Eigen values

Solution to the equations above, gives values of $w_1^2, w_2^2, \dots, w_n^2$ where $w_1, w_2, \dots, w_n$ are the nat Frequencies of the system so that $w_1 \leq w_2 \leq \dots \leq w_n$. w_1 is the Fundamental nat Frequency of the System.

- For each natural Frequency w_i, we will have a corresponding $\{X\}^{(i)}$ Such as

$$\{X\}^{(i)} = \begin{bmatrix} X_1^{(1)} & X_1^{(2)} & \cdots & X_1^{(n)} \\ X_2^{(1)} & X_2^{(2)} & \cdots & X_2^{(n)} \\ \vdots & \vdots & & \\ X_n^{(1)} & X_n^{(2)} & \cdots & X_3^{(n)} \end{bmatrix}_{n\times n}$$

← Mode shapes (Eigenvectors)

Modeshape (1) — Mode shape (2) — ... — Modeshape (n)

$$\{X\}^{(i)} = \left[\{X\}^{(1)} \; \{X\}^{(2)} \cdots \{X\}^{(n)}\right] = [U]$$

Modal matrix

Cont'd

therefore,

$$\{x(t)\} = [U]\,\eta_i(t) \quad , \quad \eta_i(t) = \sin(\omega_i t + \phi_i) \quad , \quad \{\eta\} = \eta_i(t) = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{Bmatrix}$$

eta

$$\eta_i(t) = A_i \cos\omega_i t + B_i \sin\omega_i t$$

2.2.3 Free Vibration of Undamped Systems Using Modal Analysis?

Direct method } we have two Solution methods.

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

$$\{x(t)\} = [U]\,\eta_i(t) \quad , \quad \{\ddot{x}\} = [U]\,\ddot{\eta}_i(t)$$

$$[m][U]\eta_i + [k][U]\ddot{\eta}_i = \{0\}$$

Premultiply by $[U]^T$ ← traspose

$$[U]^T[m][U]\ddot{\eta} + [U]^T[k][U]\eta_i = \{0\}$$

$[U]$ can be normalized to be $[U_n]$ such that

$$[U_n]^T[m][U_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}_{n\times n} = [I]$$

and

$$[U_n]^T[k][U_n] = \begin{bmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & & \\ \vdots & & \ddots & \\ 0 & & & \omega_n \end{bmatrix}_{n\times n} = [\omega_i^2] \quad i=1,\dots,n$$

So,

$$[U_n]^T[m][U_n]\ddot{\eta}_i + [U_n]^T[k][U_n]\eta_i = \{0\}$$

$$\ddot{\eta}_i + [\omega_i^2]\eta_i = \{0\}$$

$$\frac{d^2\eta_i}{dt^2} + \omega_i^2\eta_i = 0 \qquad i=1,2,\dots,n$$

Cont'd

For Initial conditions

$$\{x(0)\} = x_i(0) = \begin{Bmatrix} x_{1,0} \\ x_{2,0} \\ \vdots \\ x_{n,0} \end{Bmatrix} \quad \text{and} \quad \{\dot{x}(0)\} = \dot{x}_i(0) = \begin{Bmatrix} v_{1,0} \\ v_{2,0} \\ \vdots \\ v_{n,0} \end{Bmatrix}$$

we can find $\eta_i(t) = \underbrace{A_i \cos\omega_i t + B_i \sin\omega_i t}$

Initial conditions.

But remember

$$\{x(t)\} = [U]\,\eta_i \Rightarrow \eta_i = [U]^{-1}\{x(t)\}$$

now we can apply initial conditions.

or

$$\{x(t)\} = [U]\,\eta_i$$

Premultiply by $[U]^T[m]$

$$[U]^T[m]\{x(t)\} = [U]^T[m][U]\,\eta_i$$

If $[U]$ is normalized $[U_n]$

$$[U_n]^T[m]\{x(t)\} = \underbrace{[U_n]^T[m][U_n]}_{[I]}\,\eta_i$$

So,

$$\eta_i(t) = [U_n]^T[m]\{x(t)\}$$

$$\dot{\eta}_i(t) = [U_n]^T[m]\{\dot{x}(t)\}$$

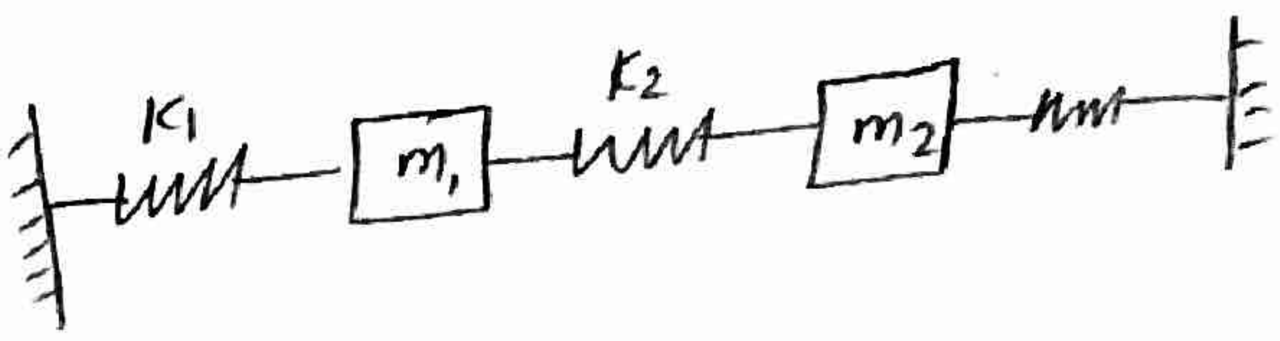
Example

$m_1 = 10\,kg$
$m_2 = 1\,kg$
$K_1 = 30\,N/m$
$K_2 = 5\,N/m$, $K_3 = 0$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2+K_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Initial conditions

$$\{x(0)\} = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad , \quad \{\dot{x}(0)\} = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



Solution

$$x(t) = \{X\} \sin(\omega t + \phi)$$

$$\Rightarrow \quad \left[[k] - \omega^2 [m]\right] \{X\} = \{0\}$$

- $$\left|[k] - \omega^2[m]\right| = 0 \Rightarrow \begin{vmatrix} K_1+K_2-\omega^2 m_1 & -K_1 \\ -K_1 & K_2+K_3-\omega^2 m_2 \end{vmatrix} = \begin{vmatrix} 35-\omega^2 10 & -5 \\ -5 & 5-\omega^2 \end{vmatrix}$$

$$\Rightarrow (35 - 10\omega^2)(5-\omega^2) - 25 = 0$$
$$10\omega^4 - 85\omega^2 + 150 = 0 \Rightarrow \begin{matrix} \omega_1^2 = 2.5 \Rightarrow \omega_1 = 1.5811 \text{ rad/sec} \\ \omega_2^2 = 6 \Rightarrow \omega_2 = 2.4495 \text{ rad/sec} \end{matrix}$$

For $\omega = \omega_1 = 1.5811$

- $$\left[[K] - \omega_1^2 [m]\right]\{X^{(1)}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 10 & -5 \\ -5 & 2.5 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left.\begin{matrix} 10X_1^{(1)} - 5X_2^{(1)} = 0 \\ -5X_1^{(1)} + 2.5X_2^{(1)} = 0 \end{matrix}\right\} \text{Same Equation}$$

$$2X_1^{(1)} = X_2^{(1)}$$

So, we assume $X_1^{(1)} = 1 \Rightarrow X_2^{(1)} = 2$

$$\{X^{(1)}\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

For $\omega = \omega_2^2 = 6$

$$\left[[k] - \omega^2[m]\right]\{X^{(2)}\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 35-60 & -5 \\ -5 & 5-6 \end{bmatrix} \Rightarrow \begin{bmatrix} -25 & -5 \\ -5 & -1 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left.\begin{matrix} -25X_1^{(2)} - 5X_2^{(2)} = 0 \\ -5X_1^{(2)} - X_2 = 0 \end{matrix}\right\} \text{Same Equation, so assume } \left.\begin{matrix} X_1^{(2)} = 1 \\ X_2^{(2)} = -5 \end{matrix}\right\} \{X^{(2)}\} = \begin{Bmatrix} 1 \\ -5 \end{Bmatrix}$$

$$5X_1^{(2)} = -X_2^{(2)}$$

cont'd

Modal matrix

$$[U] = [\{X^{(1)}\} \quad \{X^{(2)}\}] = \begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix}$$

Therefore

$$x(t) = [U]\,\eta_i(t) \quad , \quad \eta_i = \sin(\omega_i t + \phi_i) = A_i \cos\omega_i t + B_i \sin\omega_i t$$

we need to normalize $[U]$, so that: $[U_n]^T [m][U_n] = [I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In general, If

$$[U]^T[m][U] = \begin{bmatrix} a & & \phi \\ & b & \\ \phi & & c \ddots \\ & & & z \end{bmatrix} \Rightarrow [U_n] = \begin{bmatrix} \frac{U_{11}}{\sqrt{a}} & \frac{U_{12}}{\sqrt{b}} & \cdots & \frac{U_{1n}}{\sqrt{z}} \\ \frac{U_{21}}{\sqrt{a}} & \frac{U_{22}}{\sqrt{b}} & \cdots & \frac{U_{2n}}{\sqrt{z}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{U_{n1}}{\sqrt{a}} & \frac{U_{n2}}{\sqrt{b}} & \cdots & \frac{U_{nn}}{\sqrt{z}} \end{bmatrix}$$

- $$[U]^T[m][U] = \begin{bmatrix} 1 & 2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 \\ 0 & 35 \end{bmatrix}$$

- $$\Rightarrow [U_n] = \begin{bmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{-5}{\sqrt{35}} \end{bmatrix} = \begin{bmatrix} 0.2673 & 0.1690 \\ 0.5343 & -0.8452 \end{bmatrix}$$

Check

- $[U_n]^T [m][U_n] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $[U_n]^T [k][U_n] = \begin{bmatrix} 2.5 & 0 \\ 0 & 6 \end{bmatrix}$

MATLAB program is on the Moodle

Cont'd

Now, we need to find $\eta_i(t)$

$$\eta_1(t) = A_1 \cos\omega_1 t + B_1 \sin\omega_1 t$$
$$\eta_2(t) = A_2 \cos\omega_2 t + B_2 \sin\omega_2 t$$

$$\eta_i(t) = \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = [U_n]^T [m] \{x(t)\}$$

$$\dot{\eta}_i(t) = \begin{Bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{Bmatrix} = [U_n]^T [m] \{\dot{x}(t)\}$$

For Initial conditions $\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

$$\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} = \begin{bmatrix} 0.2673 & 0.5343 \\ 0.1690 & -0.8452 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} = \begin{Bmatrix} 2.6726 \\ 1.6903 \end{Bmatrix}$$

$$\eta_1(0) = A_1 \cos\omega_1(0) + \cancel{B_1 \sin\omega_1(0)} \Rightarrow A_1 = 2.6726$$
$$\eta_2(0) = A_2 \cos\omega_2(0) + \cancel{B_2 \sin\omega_1(0)}^{0} \Rightarrow A_2 = 1.6903$$

For $\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$$\begin{Bmatrix} \dot{\eta}_1(0) \\ \dot{\eta}_2(0) \end{Bmatrix} = \begin{bmatrix} 0.2673 & 0.5343 \\ 0.1690 & -0.8452 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\eta}_1(0) \\ \dot{\eta}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\dot{\eta}_1(t) = -\omega_1 A_1 \sin\omega_1 t + \omega_1 B_1 \cos\omega_1 t$$
$$\dot{\eta}_2(t) = -\omega_2 A_2 \sin\omega_2 t + \omega_2 B_2 \cos\omega_2 t$$

$$\dot{\eta}_1(0) = \omega_1 B_1 = 0 \Rightarrow B_1 = 0$$
$$\dot{\eta}_2(0) = \omega_2 B_2 = 0 \Rightarrow B_2 = 0$$

$$\eta_1(t) = 2.6726 \cos\omega_1 t$$
$$\eta_2(t) = 1.6903 \cos\omega_1 t$$

$$x(t) = \begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix}$$

cont'd

$$\{x(t)\} = \begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix}$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} \eta_1(t) + \eta_2(t) \\ 2\eta_1(t) - 5\eta_2(t) \end{Bmatrix} = \begin{Bmatrix} 2.6726 \cos \omega_1 t + 1.6903 \cos \omega_2 t \\ 5.3453 \cos \omega_1 t - 8.4515 \cos \omega_2 t \end{Bmatrix} \begin{matrix} \leftarrow x_1(t) \\ \leftarrow x_2(t) \end{matrix}$$

- we can also use

$$x(t) = [U_n] \eta_i(t)$$

For practice, Solve Example 2.2 → textbook 1

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