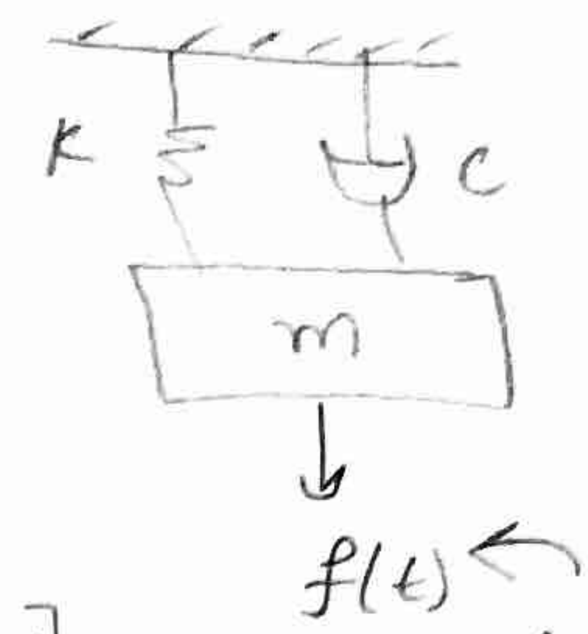
2.1.3 Forced Vibration under General Force

$x(0) = x_0$
$\dot{x}(0) = v_0$

EOM

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = F(t) \quad , \; F(t) = \frac{f(t)}{m}$$



Any force

Take Laplace

$$\mathcal{L}[\ddot{x}(t)] + 2\zeta\omega_n \mathcal{L}[\dot{x}(t)] + \omega_n^2 \mathcal{L}[x(t)] = \mathcal{L}[F(t)]$$

Remember

$\mathcal{L}[x(t)] = X(s)$

$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0)$

$\mathcal{L}[\ddot{x}(t)] = s^2 X(s) - s x(0) - \dot{x}(0)$

$\mathcal{L}[F(t)] = F(s)$

$$\Rightarrow \left[s^2 X(s) - s x(0) - \dot{x}(0)\right] + 2\zeta\omega_n\left[s X(s) - x(0)\right] + \omega_n^2 X(s) = F(s)$$

$$X(s)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) - x(0)\left(s + 2\zeta\omega_n\right) - \dot{x}(0) = F(s)$$

$$X(s) = \underbrace{\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}}_{= H(s)} \left[F(s) + x_0\left(s + 2\zeta\omega_n\right) + v_0\right]$$

$$X(s) = H(s)\, F(s) + \left(H(s)\right) x_0 (s + 2\zeta\omega_n) + H(s)\, v_0$$

Take Laplace Inverse $\mathcal{L}^{-1}[\;]$

$$\mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}[H(s) F(s)] + \mathcal{L}^{-1}[H(s)\, x_0 (s + 2\zeta\omega_n)] + v_0 \mathcal{L}^{-1}[H(s)]$$

$$x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau + g(t)\, x_0 + h(t)\, v_0$$

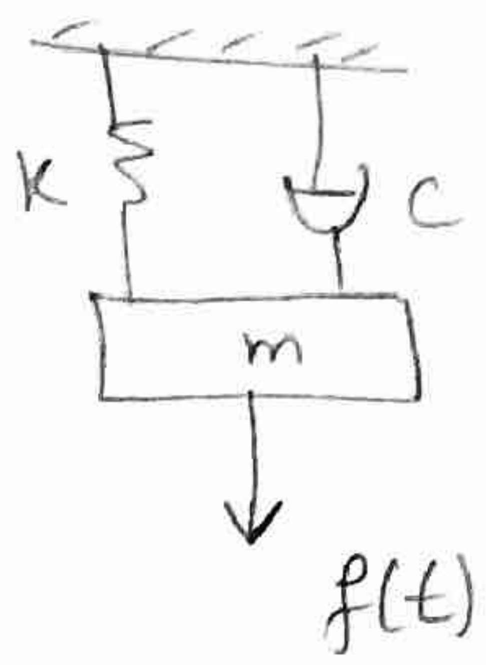
$$g(t) = e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right) , \quad h(t) = \frac{1}{\omega_d} e^{-\zeta\omega_n t}\left(\sin\omega_d t\right)$$

Example 1

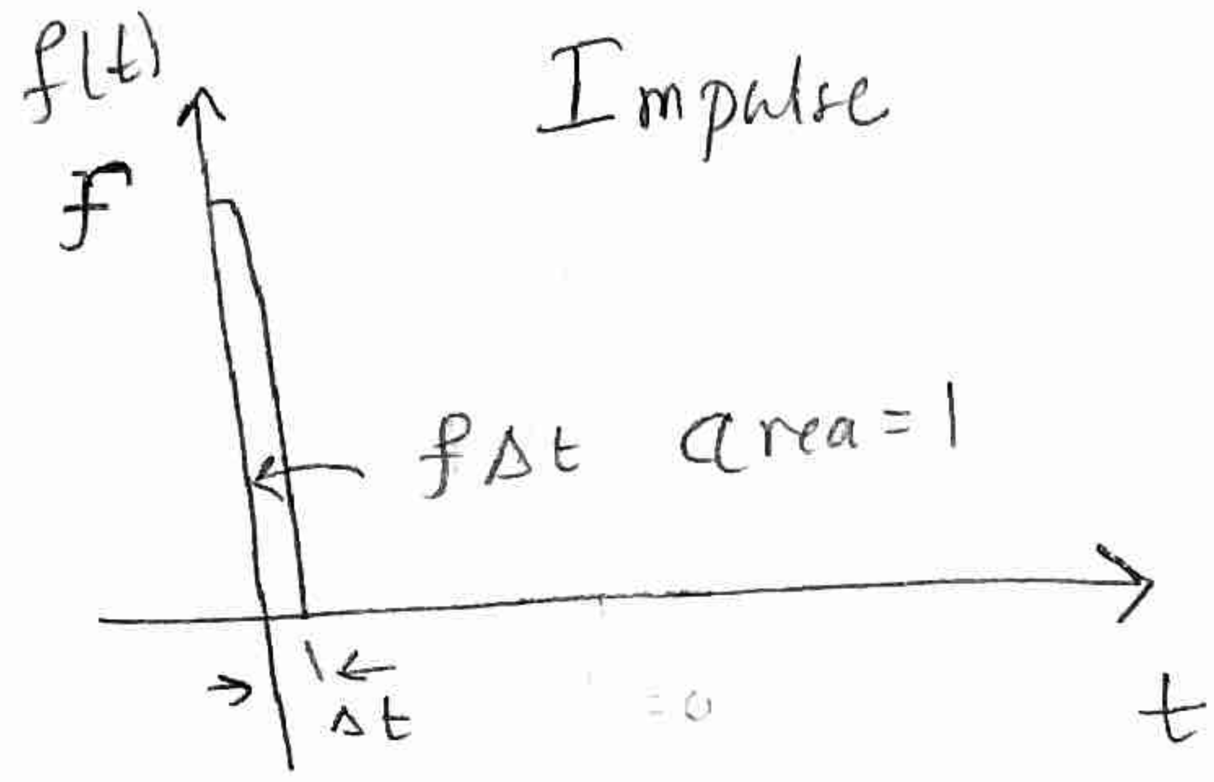
$x(0)=0$

$\dot{x}(0)=0$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t)$$



$$f(t) = \begin{cases} \int_{-0^-}^{0^+} f(t) = 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$



$\Delta t = 0^- \to 0^+$ ← very short

the variable is τ

$$x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau = \int_{0^-}^{0^+} \frac{f(\tau)}{m}\, h(t-0)\, d\tau$$

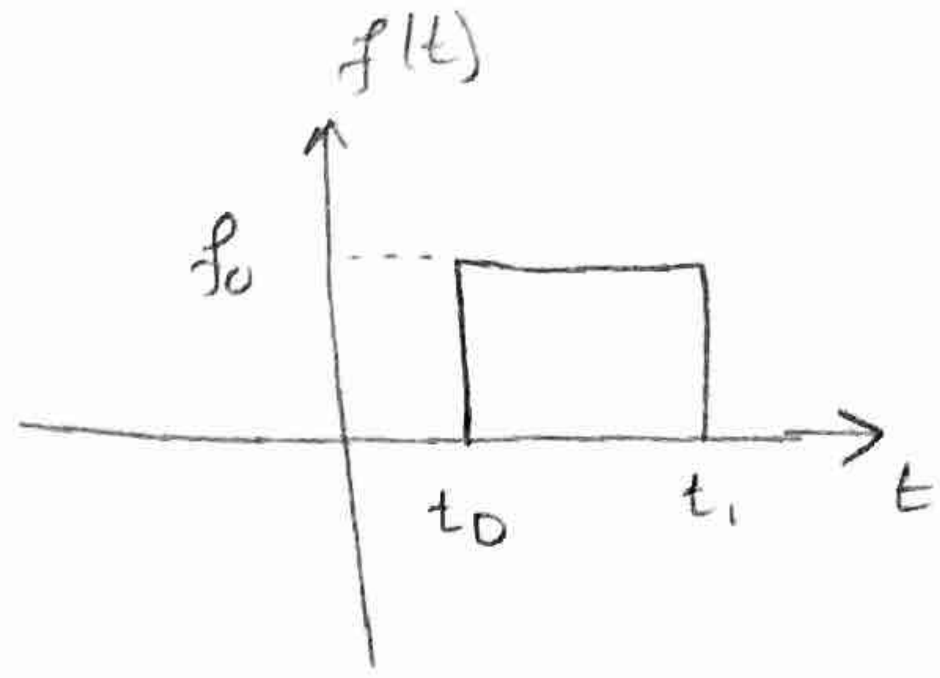
$$= \frac{h(t)}{m} \int_{0^-}^{0^+} \underbrace{f(\tau)\, d\tau}_{=1} \Rightarrow x(t) = \frac{h(t)}{m}$$

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t)$$ ← Response to Impulse function

Example 2

$x(0)=0$
$\dot{x}(0)=0$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = F(t) \quad , \; F(t) = \frac{f(t)}{m}$$



$$f(t) = \begin{cases} 0 & 0 \le t < t_0 \\ f_0 & t_0 \le t \le t_1 \\ 0 & t > t_1 \end{cases}$$

Find $x(t)$ in terms of Convolution Integral

$0 \le t < t_1$

$$x(t) = \int_0^t \cancel{\frac{f(\tau)}{m}}^{\,0} h(t-\tau)\, d\tau \;\Rightarrow\; x(t) = 0$$

$t_0 \le t < t_1$

$$x(t) = \int_0^{t_0} \cancel{\frac{f(\tau)}{m}}^{\,0} \cdot h(t-\tau)\, d\tau + \int_{t_0}^{t} \frac{f(\tau)}{m} \cdot h(t-\tau)\, d\tau$$

$$x(t) = \int_{t_0}^{t} \frac{f_0}{m} \cdot \frac{1}{\omega_d} e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

$$= \frac{f_0}{m\omega_d} \int_{t_0}^{t} e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

$t > t_1$

$$x(t) = \int_0^{t_0} \cancel{\frac{f(\tau)}{m}}^{\,0} h(t-\tau)\, d\tau + \int_{t_0}^{t_1} \frac{f(\tau)}{m} h(t-\tau)\, d\tau + \int_{t_1}^{t} \cancel{\frac{f(\tau)}{m}}^{\,0} \cdot h(t-\tau)\, d\tau$$

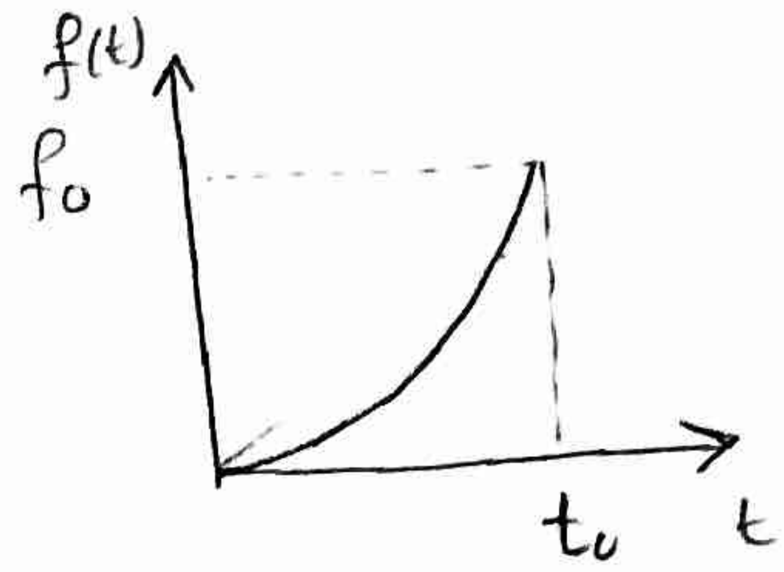
$$x(t) = \frac{f_0}{m\omega_d} \int_{t_0}^{t_1} e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$$

Example (3)

$x(0) = 0$
$\dot{x}(0) = 0$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t) \quad , \quad F(t) = \frac{f(t)}{m}$$

$$f(t) = \begin{cases} f_0 \dfrac{t}{t_0} & 0 \leq t \leq t_0 \\ 0 & t > t_0 \end{cases}$$



Find $x(t)$ in terms of convolution Integral

Solution

$0 \leq t \leq t_1$

$$x(t) = \int_0^t F(\tau)h(t-\tau)\,d\tau = \int_0^t \frac{f_0}{m}\frac{\tau}{t_0} \cdot h(t-\tau)\,d\tau$$

$$x(t) = \frac{f_0}{m t_0 \omega_d} \int_0^t \tau \, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\,d\tau$$

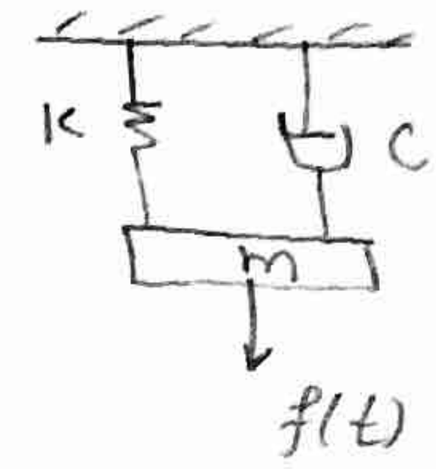
$t \geq t_1$

$$x(t) = \int_0^{t_1} F(\tau)h(t-\tau)\,d\tau + \int_{t_1}^{t} \cancel{F(\tau)}^{0} h(t-\tau)\,d\tau$$

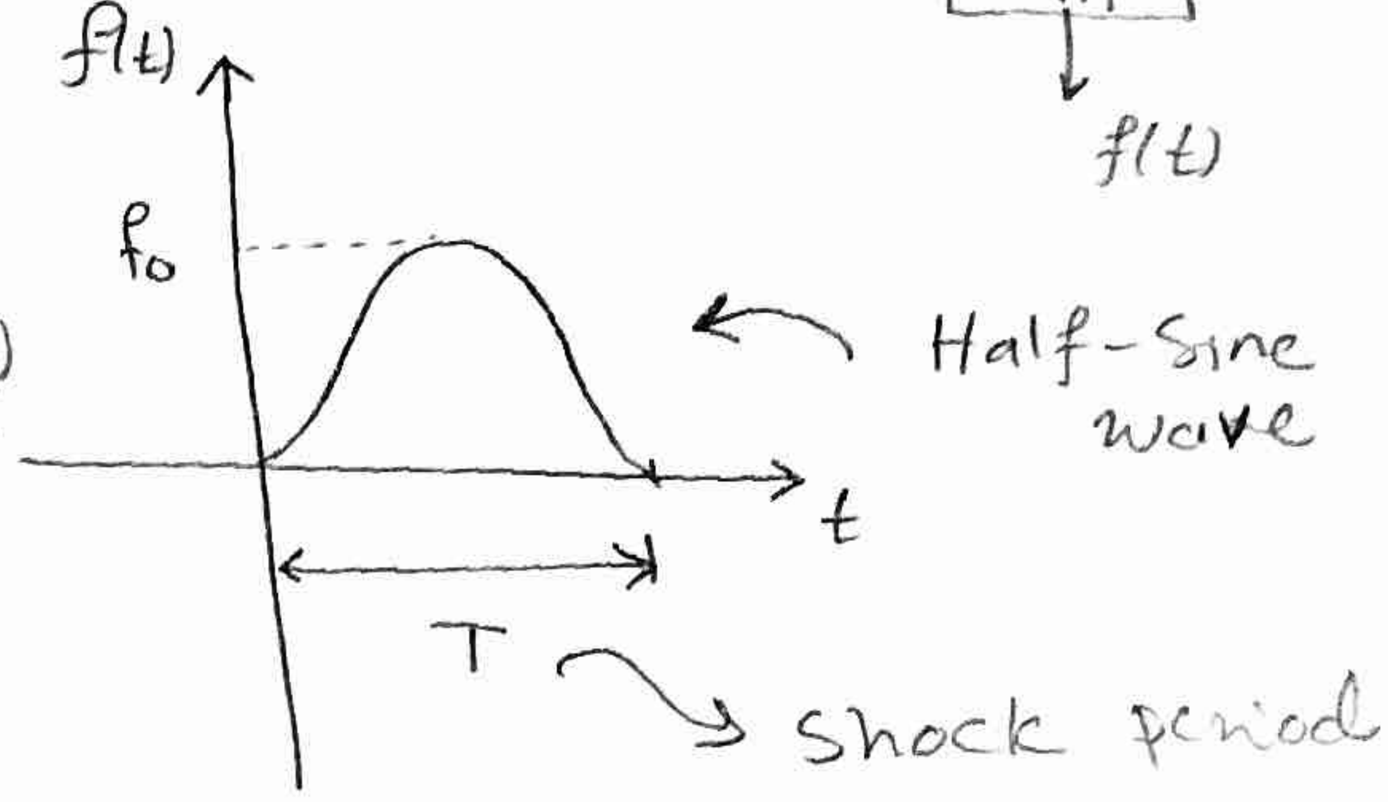
$$x(t) = \frac{f_0}{m t_0 \omega_d} \int_0^{t_1} \tau \, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\,d\tau$$

* Example ④

$x(0)=0$
$\dot{x}(0)=0$



$$f(t) = \begin{cases} f_0 \sin\left(\frac{\pi t}{T}\right) & 0 \le t \le T \\ 0 & t > T \end{cases}$$



EOM

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = F(t), \quad F(t) = \frac{f(t)}{m}$$

Express Response ($x(t)$) in terms of Convolution Integral

Solution

$$x(t) = \int_0^t F(\tau)\, h(t-\tau)\, d\tau + g(x)\, \cancel{x_0}^{\,0} + h(t)\, \cancel{v_0}^{\,0}$$

$0 \le t \le T$

$$x(t) = \int_0^t \frac{f_0}{m} \sin\left(\frac{\pi \tau}{T}\right) h(t-\tau)\, d\tau = \frac{f_0}{m} \int_0^t \sin\left(\frac{\pi \tau}{T}\right) h(t-\tau)\, d\tau$$

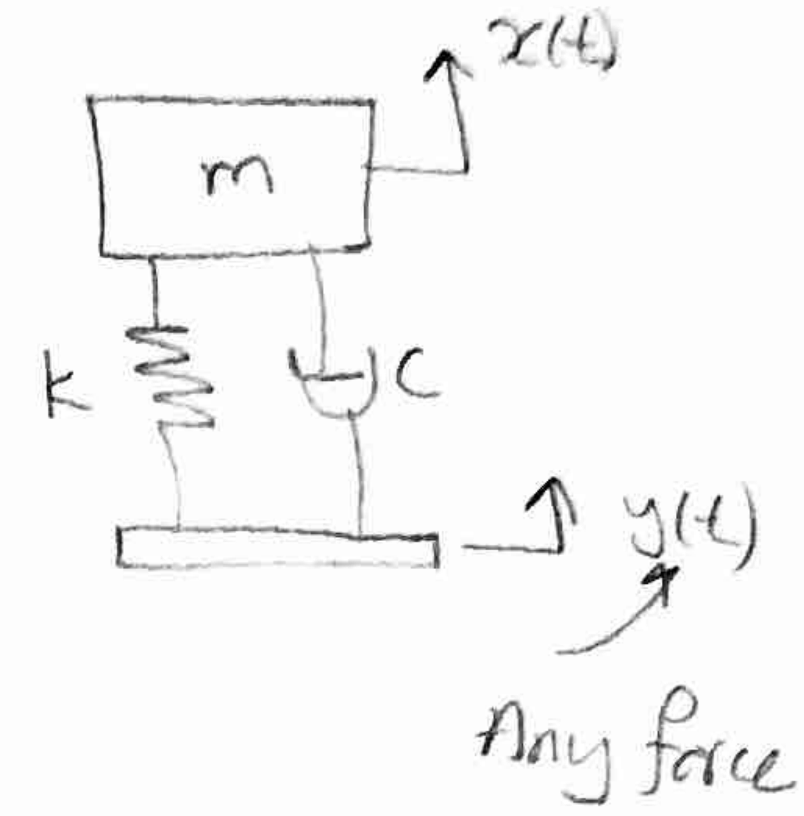
$t \ge T$

$$x(t) = \int_0^T \frac{f_0}{m} \sin\left(\frac{\pi \tau}{T}\right) h(t-\tau)\, d\tau + \int_T^t \cancel{F(\tau)}^{\,0}\, h(t-\tau)\, d\tau$$

$$= \frac{f_0}{m} \int_0^T \sin\left(\frac{\pi \tau}{T}\right) h(t-\tau)\, d\tau$$

Example ⑤ : Response due to Base Excitation

$x(0) = x_0$
$\dot{x}(0) = v_0$



EOM

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t)$$

Solution method ①

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = F(t), \quad F(t) = 2\zeta\omega_n \dot{y} + \omega_n^2 y$$

$$x(t) = \int_0^t F(\tau) h(t-\tau)\, d\tau + g(t) x_0 + h(t) v_0$$

$$= \int_0^t \left(2\zeta\omega_n \dot{y}(\tau) + \omega_n^2 y(\tau)\right) h(t-\tau)\, d\tau + g(t) x_0 + h(t) v_0$$

$$= 2\zeta\omega_n \int_0^t \dot{y}(\tau) h(t-\tau)\, d\tau + \omega_n^2 \int_0^t y(\tau) h(t-\tau)\, d\tau + g(t) x_0 + h(t) v_0$$

Solution method ②

$$z(t) = x(t) - y(t) \quad \Rightarrow \quad x(t) = z(t) + y(t)$$

$$\ddot{z}(t) + 2\zeta\omega_n \dot{z} + \omega_n^2 z = F(t), \quad F(t) = -\ddot{y}(t)$$

$$z(t) = \int_0^t F(\tau) h(t-\tau)\, d\tau + g(t) x_0 + h(t) v_0$$

$$z(t) = -\int_0^t \ddot{y}(\tau) h(t-\tau)\, d\tau + g(t) x_0 + h(t) v_0$$

$$\Rightarrow \quad x(t) = y(t) - \left[\int_0^t \ddot{y}(\tau) h(t-\tau)\, d\tau + g(t) x_0 + v_0 h(t)\right]$$