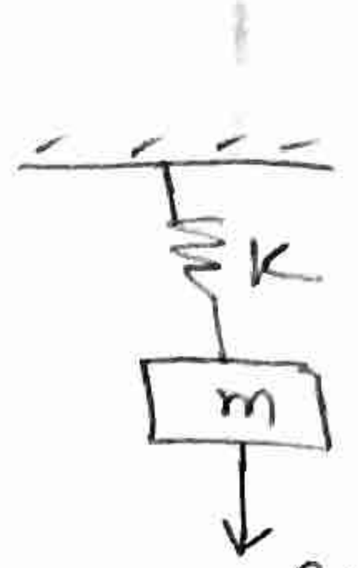
(3)

* Complex notation f. Harmonic excitation

(1) Undamped systems



$f(t) = f_0 \, e^{i\omega t}$

$$m\ddot{x} + Kx = f(t)$$

$$\ddot{x} + \omega_n^2 x = F(t) \quad , \; F(t) = \frac{f(t)}{m}$$

$$F(t) = F_0 e^{i\omega t} \qquad \frac{f_0}{m} = F_0$$

$$x = X e^{i\omega t}$$
$$\dot{x} = i\omega X e^{i\omega t}$$
$$\ddot{x} = -\omega^2 X e^{i\omega t}$$

$$-\omega^2 X e^{i\omega t} + \omega_n^2 X e^{i\omega t} = F_0 e^{i\omega t}$$

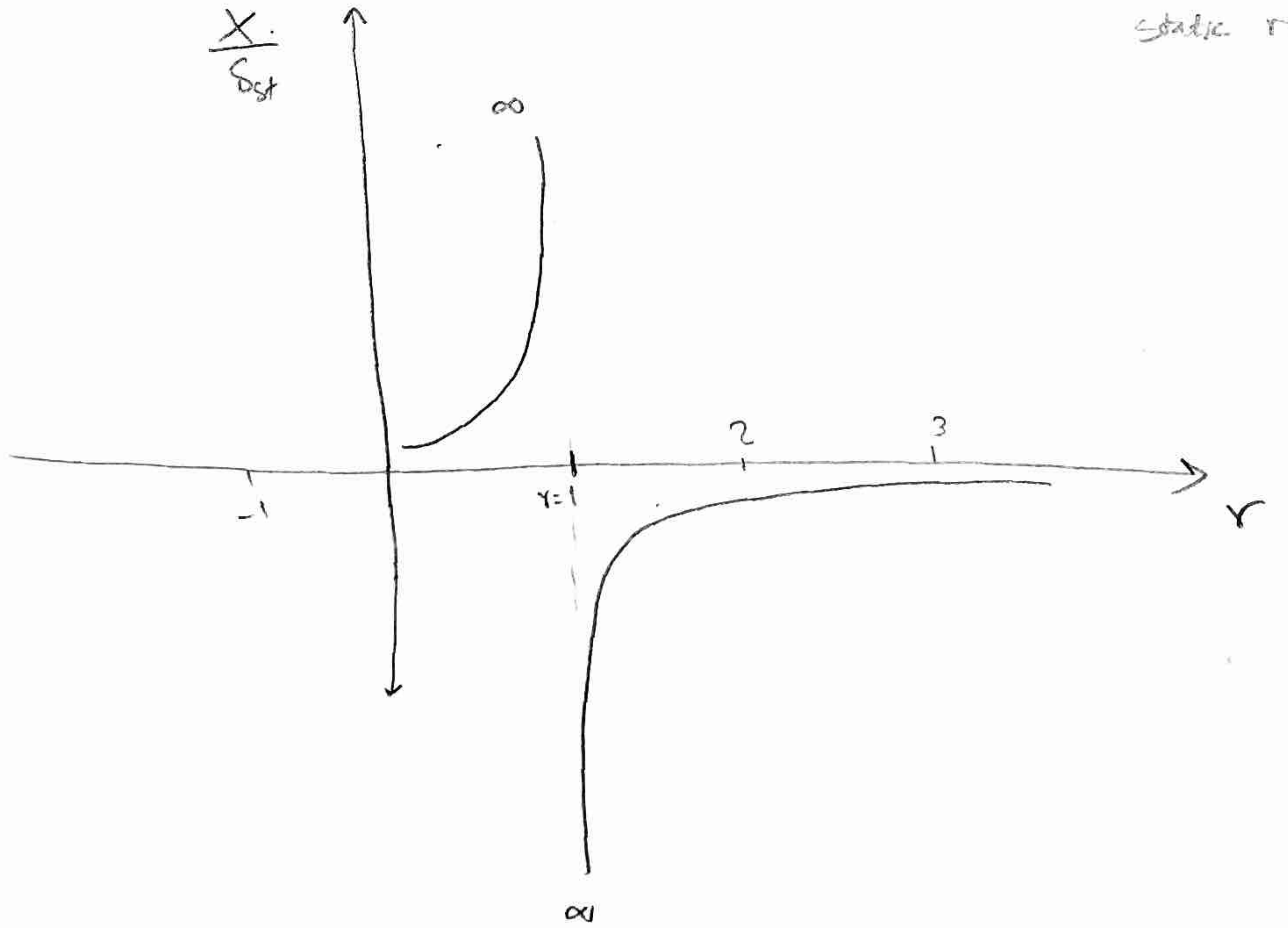
$$X(\omega_n^2 - \omega^2) = F_0 \quad \Rightarrow \quad X = \frac{F_0}{\omega_n^2 - \omega^2}$$

or $$X = \frac{F_0}{\omega_n^2\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)} = \frac{F_0/\omega_n^2}{(1 - r^2)} \quad , \quad r = \frac{\omega}{\omega_n}$$

$$\frac{F_0}{\omega_n^2} = \frac{f_0}{m \frac{k}{m}} = \frac{f_0}{k} = \delta_{static}$$

$$\Rightarrow X = \frac{\delta_{static}}{1 - r^2} \quad \Rightarrow \quad \frac{X}{\delta_{static}} = \frac{1}{1 - r^2}$$

→ ratio between dynamic (X) and static responses, amplification factor



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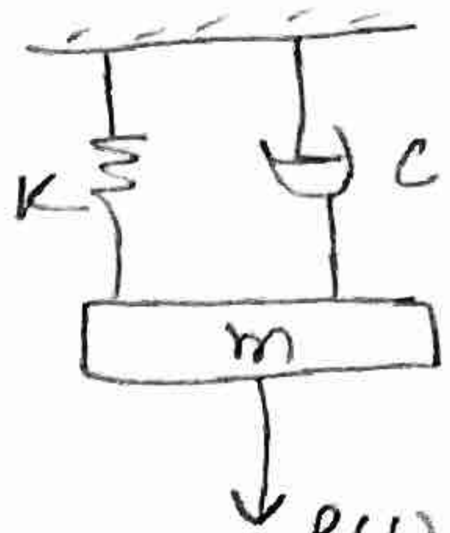
② Damped systems

$$F(t) = \frac{f(t)}{m}$$
$$F(t) = F_o e^{\hat{i}\omega t}$$
$$F_o = \frac{f_o}{m}$$



$f(t) = f_o e^{\hat{i}\omega t}$

$$m\ddot{x} + c\dot{x} + Kx = f(t)$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = F(t)$$

$$x(t) = X e^{i\omega t}, \quad \dot{x} = \hat{i}\omega X e^{i\omega t}, \quad \ddot{x} = -\omega^2 X e^{i\omega t}$$

$$-\omega^2 X e^{i\omega t} + \hat{i}2\zeta\omega_n\omega X e^{i\omega t} + \omega_n^2 X e^{i\omega t} = F_o e^{i\omega t}$$

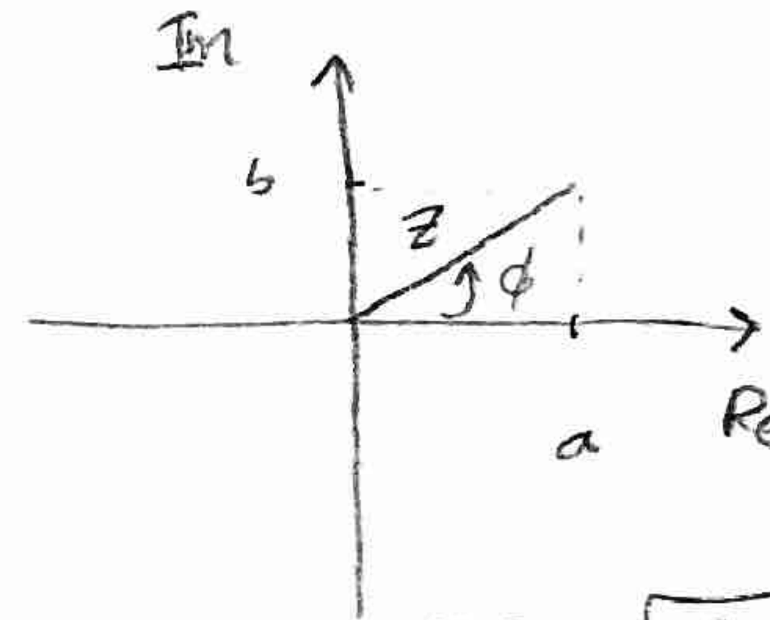
$$X = \frac{F_o}{(\omega_n^2 - \omega^2) + 2\hat{i}\zeta\omega_n\omega}$$ → complex number
- → Magnitude
- → phase

→ $X = |X| e^{i\phi}$ → phase
(Amp → $|X|$)

$$X = \frac{F_o}{(\omega_n^2 - \omega^2) + 2\hat{i}\zeta\omega_n\omega} \cdot \frac{(\omega_n^2 - \omega^2) - 2\hat{i}\zeta\omega_n\omega}{(\omega_n^2 - \omega^2) - 2\hat{i}\zeta\omega_n\omega}$$

$z = a + \hat{i}b$ (a → real, b → Imaginary)

$z = |Z| e^{i\phi}$ ($|Z|$ → mag., ϕ → phase)



$$X = \frac{F_o\left((\omega_n^2 - \omega^2) - 2\hat{i}\zeta\omega_n\omega\right)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$|Z| = \sqrt{a^2 + b^2} = \sqrt{Re^2 + Im^2}$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$ (b → Im, a → Re)

$$X = \underbrace{\frac{F_o(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}_{\text{Real}} - \underbrace{\frac{F_o(2\hat{i}\zeta\omega_n\omega)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}_{\text{Imaginary}}$$

$$X = \sqrt{Re^2 + Im^2} \Rightarrow X = \frac{F_o}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \Rightarrow \frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

($\frac{X}{\delta_{st}}$ → Amp ratio)

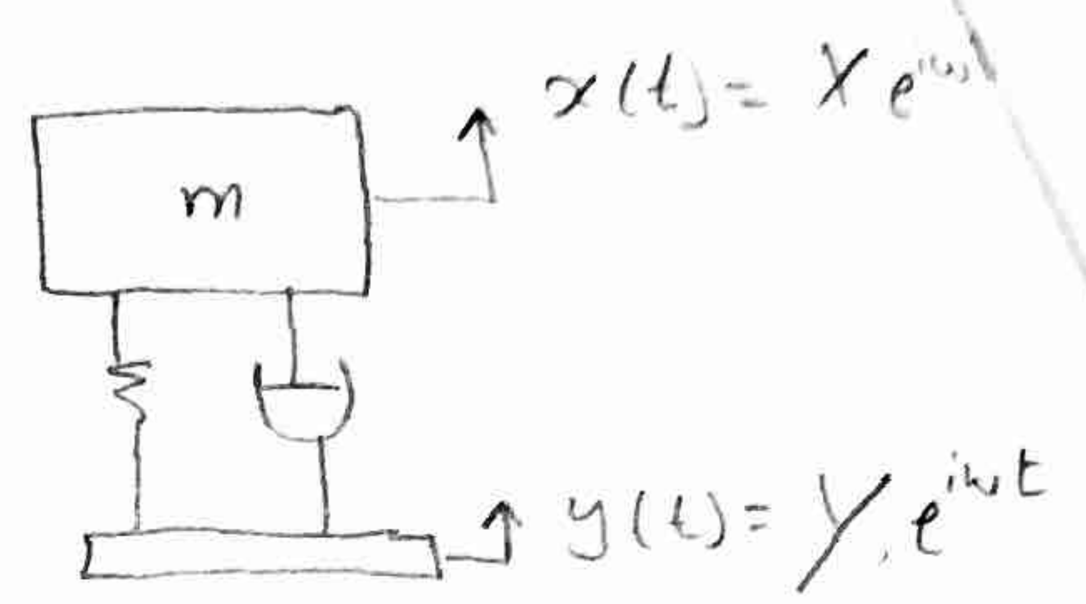
$$r = \frac{\omega}{\omega_n}$$

$$\phi = \tan^{-1}\left(\frac{Im}{Re}\right) \qquad \phi = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

For plots see text book 1 Page 40 Fig 2.6

Cont'd

Base Excitation



$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 2\zeta\omega_n \dot{y} + \omega_n^2 y$$

$$-\omega^2 X e^{i\omega t} + 2i\zeta\omega_n\omega X e^{i\omega t} + \omega_n^2 X e^{i\omega t} = 2i\zeta\omega_n\omega Y e^{i\omega t} + \omega_n^2 Y e^{i\omega t}$$

$$X = \frac{Y\left(\omega_n^2 + 2i\zeta\omega_n\omega\right)}{(\omega_n^2 - \omega^2) + (2i\zeta\omega_n\omega)}$$

$$X = |X| e^{i\phi}$$

- $$|X| = Y\left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)$$

- $$\frac{|X|}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

→ transmissibility $\boxed{T_d = \frac{X}{Y}}$

See Figure 3.15, page 283 } textbook ≥