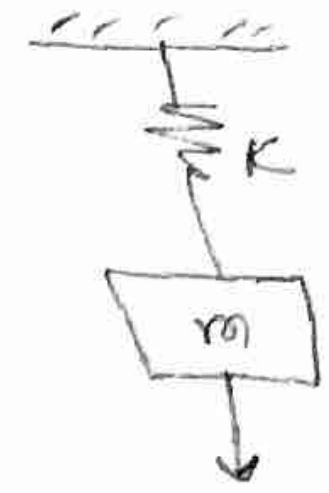
(2)

2.1.2 Forced vibration (Harmonic excitation)

(1) undamped systems

$x(0) = x_0$

$\dot{x}(0) = v_0$



$f(t) = f_0 \cos \omega t$

ω: Excitation Frequency rad/sec

EOM

$$m\ddot{x} + kx = f(t) \qquad \frac{f(t)}{m} = F(t)$$

$$\ddot{x} + \omega_n^2 x = F(t) \qquad -(1) \qquad F(t) = F_0 \cos\omega t$$

$$F_0 = \frac{f_0}{m}$$

Non-homogenuous Diff. Eq.

$$x(t) = x_h(t) + x_p(t)$$

homog. solution ($m\ddot{x} + kx = 0$) — particular solution

- we know

$$x_h(t) = A \cos\omega_n t + B \sin\omega_n t$$

Particular solution

$x_p(t) = X \cos\omega t$, subst. in Eq (1) // or $x_p(t) = \alpha \cos\omega t + \beta \sin\omega t$

$\alpha = \frac{F_0}{\omega_n^2 - \omega^2}$ will $\beta = 0$

$$\dot{x}_p(t) = -\omega X \sin\omega t$$

$$\ddot{x}_p(t) = -\omega^2 X \cos\omega t$$

$$-\omega^2 X \cancel{\cos\omega t} + \omega_n^2 X \cancel{\cos\omega t} = F_0 \cancel{\cos\omega t}$$

- $$X(\omega_n^2 - \omega^2) = F_0 \Rightarrow X = \frac{F_0}{\omega_n^2 - \omega^2} \Big\} \Rightarrow x_p(t) = \frac{F_0}{\omega_n^2 - \omega^2} \cos\omega t$$

Total Solution

$$x(t) = A \cos\omega_n t + B \sin\omega_n t + \frac{F_0}{\omega_n^2 - \omega^2} \cos\omega t$$

to find A and B $\Rightarrow$ IC's

$$x(0) = x_0 \Rightarrow x(0) = x_0 = A + \frac{F_0}{\omega_n^2 - \omega^2} \Rightarrow A = x_0 - \frac{F_0}{\omega_n^2 - \omega^2}$$

$\dot{x}(0) = v_0 \Rightarrow$

$$\dot{x}(t) = -\omega_n A \sin\omega_n t + \omega_n B \cos\omega_n t - \omega \frac{F_0}{\omega_n^2 - \omega^2} \sin\omega t$$

$$\dot{x}(0) = v_0 = \omega_n B \Rightarrow B = \frac{v_0}{\omega_n}$$

$$x(t) = \left(x_0 - \frac{F_0}{\omega_n^2 - \omega^2}\right) \cos\omega_n t + \frac{v_0}{\omega_n} \sin\omega_n t + \frac{F_0}{\omega_n^2 - \omega^2} \cos\omega t$$

(2) Damped systems

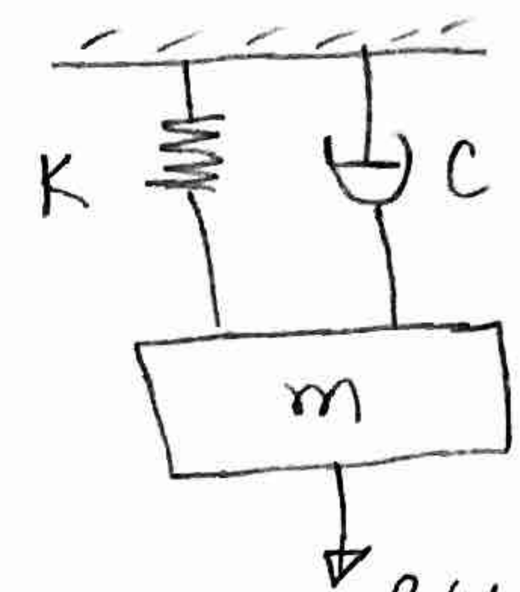
EOM

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\frac{f(t)}{m} = F(t)$$

$$\frac{f_0}{m} = F_0$$

$$F(t) = F_0 \cos\omega t$$



$$f(t) = f_0 \cos\omega t$$

$x(0) = x_0$,

$\dot{x}(0) = v_0$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = F(t)$$

$\hookrightarrow$ Non- Homog. ODE

$$x(t) = x_h(t) + x_p(t)$$

Particular Solution

$$x_p(t) = A_p \cos\omega t + B_p \sin\omega t$$

$$\dot{x}_p(t) = -\omega A_p \sin\omega t + \omega B_p \cos\omega t$$

$$\ddot{x}_p(t) = -\omega^2 A_p \cos\omega t - \omega^2 B_p \sin\omega t$$

$$-\omega^2 A_p \cos\omega t - \omega^2 B_p \sin\omega t - 2\zeta\omega_n\omega A_p \sin\omega t + 2\zeta\omega_n\omega B_p \cos\omega t$$

$$+ \omega_n^2 A_p \cos\omega t + \omega_n^2 B_p \sin\omega t = 0$$

$$\cos\omega t\left(-\omega^2 A_p + 2\zeta\omega_n\omega B_p + \omega_n^2 A_p\right) + \sin\omega t\left(-\omega^2 B_p - 2\zeta\omega_n\omega A_p + \omega_n^2\right) = F_0 \cos\omega t$$

$$(\omega_n^2 - \omega^2) A_p + 2\zeta\omega_n\omega B_p = F_0$$

$$-2\zeta\omega_n\omega A_p + (\omega_n^2 - \omega^2) B_p = 0$$

$$\begin{bmatrix} \omega_n^2 - \omega^2 & 2\zeta\omega_n\omega \\ -2\zeta\omega_n\omega & \omega_n^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} A_p \\ B_p \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$A_p = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \quad , \quad B_p = \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$x_p(t) = A_p \cos\omega t + B_p \sin\omega t$$

Cont'd

- Homogenous Solution

① $\zeta = 1$

$$x_h(t) = A e^{-w_n t} + B t e^{-w_n t}$$

Total Solution

$$x(t) = A e^{-w_n t} + B t e^{-w_n t} + A_p \cos wt + B_p \sin wt$$

To Find A and B $\Rightarrow$ IC's

$$x(0) = x_0 = A + A_p \Rightarrow A = x_0 - A_p \Rightarrow \boxed{A = x_0 - \frac{(w_n^2 - w^2) f_0}{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2}}$$

$$\dot{x}(t) = -w_n A e^{-w_n t} - w_n B t e^{-w_n t} + B e^{-w_n t} - w A_p \sin wt + w B_p \cos wt$$

$$\dot{x}(0) = v_0 = -w_n A - 0 + B - 0 + w B_p$$

$$\Rightarrow B = v_0 + w_n A - w B_p$$

$$B = v_0 + w_n \left(x_0 - \frac{f_0 (w_n^2 - w^2)}{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2} \right) - \frac{2\zeta w_n w^2 f_0}{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2}$$

$\zeta < 1$

$$x_h(t) = e^{-\zeta w_n t} (A \cos w_d t + B \sin w_d t)$$

Total solution

$$x(t) = e^{-\zeta w_n t} (A \cos w_d t + B \sin w_d t) + A_p \cos wt + B_p \sin wt$$

Apply BC's

$$A = x_0 - \frac{f_0 (w_n^2 - w^2)}{(w_n^2 - w^2) + (2\zeta w_n w)^2}$$

$$B = \frac{\zeta w_n}{w_d} A + \frac{v_0}{w_d} - \frac{w B}{w_d}$$

$$B = \frac{\zeta w_n}{w_d} \left(x_0 - \frac{f_0 (w_n^2 - w^2)}{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2} \right) - \left(\frac{2\zeta w_n w^2 f_0}{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2} \right) \cdot \frac{1}{w_d} + \frac{v_0}{w_d}$$

cont'd

③ $\xi > 1$

Homog. Solution

$$x_h(t) = e^{-\xi w_n t}\left(A e^{w_n\sqrt{\xi^2-1}\,t} + B e^{-w_n\sqrt{\xi^2-1}\,t}\right)$$

Total solution

$$x(t) = e^{-\xi w_n t}\left(A e^{w_n\sqrt{\xi^2-1}\,t} + B e^{-w_n\sqrt{\xi^2-1}\,t}\right) + A_p \cos wt + B_p \sin wt$$

$$x(0) = x_0 \Rightarrow \boxed{A + B + A_p = x_0} \text{ or } \boxed{A + B = x_0 - A_p}$$

$$\dot{x}(t) = e^{-\xi w_n t}\left(A w_n\sqrt{\xi^2-1}\, e^{w_n\sqrt{\xi^2-1}\,t} - B w_n\sqrt{\xi^2-1}\, e^{-w_n\sqrt{\xi^2-1}\,t}\right)$$

$$-\xi w_n e^{-\xi w_n t}\left(A e^{w_n\sqrt{\xi^2-1}\,t} + B e^{-w_n\sqrt{\xi^2-1}\,t}\right) - wA_p \sin wt + w B_p \cos wt$$

$$\dot{x}(0) = v_0 \Rightarrow \left(A w_n\sqrt{\xi^2-1} - B w_n\sqrt{\xi^2-1}\right) - \xi w_n (A + B) + w B_p = v_0$$

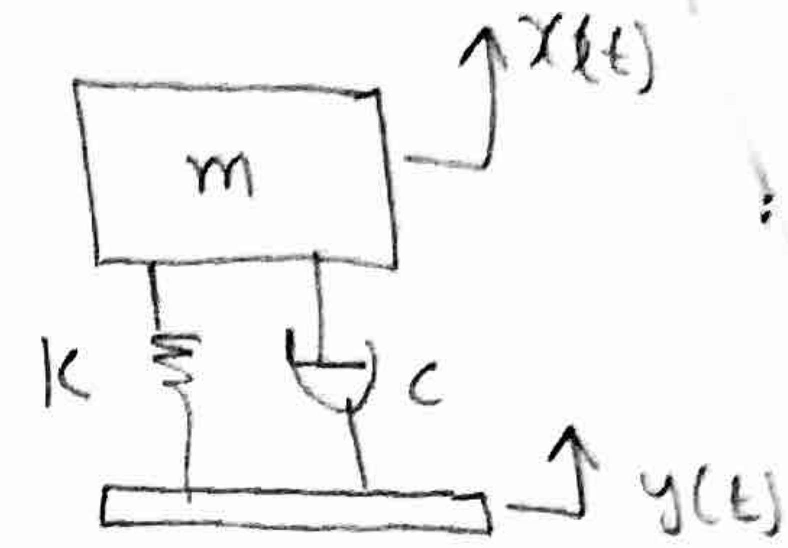
$$A\left(w_n\sqrt{\xi^2-1} - \xi w_n\right) - B\left(w_n\sqrt{\xi^2-1} + \xi w_n\right) = v_0 - w B_p$$

$$\begin{bmatrix} 1 & 1 \\ w_n\sqrt{\xi^2-1} - \xi w_n & w_n\sqrt{\xi^2-1} + \xi w_n \end{bmatrix}\begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} x_0 - A_p \\ v_0 - w B_p \end{Bmatrix}$$

$$A = \frac{1}{2\xi w_n}\left[\left(w_n\sqrt{\xi^2-1} + \xi w_n\right)(x_0 - A_p) - (v_0 - w B_p)\right]$$

$$B = \frac{1}{2\xi w_n}\left[\left(\xi w_n - w_n\sqrt{\xi^2-1}\right)(x_0 - A_p) + (v_0 - w B_p)\right]$$

Base Excitation



$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

relative motion

$$z(t) = x - y$$

$$\dot{z}(t) = \dot{x} - \dot{y}$$

$$\ddot{z}(t) = \ddot{x} - \ddot{y} \Rightarrow \ddot{x} = \ddot{z} + \ddot{y}$$

$$y(t) = Y\cos\omega_b t$$

$$\dot{y}(t) = \omega_b Y \sin\omega_b t$$

$$\ddot{y}(t) = \underbrace{-\omega_b^2 Y}_{\ddot{Y}}\cos\omega_b t$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{y} \quad , \quad \ddot{y} = \ddot{Y}\cos\omega_b t$$

$$z(t) = z_h + z_p$$

z_h ↳ As we did before

$$z_p = A_b\cos\omega_b t + B_b\sin\omega_b t$$

$$\dot{z}_p = -\omega_b A_b\sin\omega_b t + \omega_b B_b\cos\omega_b t$$

$$\ddot{z}_p = -\omega_b^2 A_b\cos\omega_b t - \omega_b^2 B_b\sin\omega_b t$$

$$-\omega_b^2 A_b\cos\omega_b t - \omega_b^2 B_p\sin\omega_b t + 2\zeta\omega_n\omega_b B_p\cos\omega_b t - 2\zeta\omega_n\omega_b A_b\sin\omega_b t$$

$$+\ \omega_n^2 A_b\cos\omega_b t + \omega_n^2 B_b\sin\omega_b t = \ddot{Y}\cos\omega_b t$$

$$\cos\omega_b\left(-\omega_b^2 A_b + 2\zeta\omega_n\omega_b B_p + \omega_n^2 A_b\right) = \ddot{Y}\cos\omega_b t$$

$$\sin\omega_b\left(-\omega_b^2 B_p - 2\zeta\omega_n\omega_b A_b + \omega_n^2 B_p\right) = 0$$

$$\begin{bmatrix} \omega_n^2-\omega_b^2 & 2\zeta\omega_n\omega_b \\ -2\zeta\omega_n\omega_b & \omega_n^2-\omega_b^2 \end{bmatrix}\begin{Bmatrix} A_b \\ B_b \end{Bmatrix} = \begin{bmatrix} \ddot{Y} \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} A_b &= \frac{\ddot{Y}(\omega_n^2-\omega^2)}{(\omega_n^2-\omega_b^2)+(2\zeta\omega_n\omega_b)^2} \\ B_b &= \frac{2\zeta\omega_n\omega_b\ddot{Y}}{(\omega_n^2-\omega_b^2)+(2\zeta\omega_n\omega_b)^2} \end{aligned}$$

Cont'd

① $\zeta = 1$

homog. soln $\Leftarrow$ $z_h = A e^{-w_n t} + B t e^{-w_n t}$

Total Solution

$$z(t) = A e^{-w_n t} + B t e^{-w_n t} + A_b \cos w_b t + B_b \sin w_b t$$

$$\Rightarrow x(t) = z(t) + y(t) = A e^{-w_n t} + B t e^{-w_n t} + A_b \cos w_b t + B_b \sin w_b t + y(t)$$

Now, we find A and B from IC's

② $\zeta < 1$

homog. $\Rightarrow z_h(t) = e^{-\zeta w_n t} \left(A \cos w_d t + B \sin w_d t \right)$

Total $\Rightarrow z(t) = e^{-\zeta w_n t} \left(A \cos w_d t + B \sin w_d t \right) + A_b \cos w_d t + B_b \sin w_b t$

$$\Rightarrow x(t) = e^{-\zeta w_n t} \left(A \cos w_d t + B \sin w_d t \right) + A_b \cos w_d t + B_b \sin w_b t + y(t)$$

Now, we get A and B from IC's

③ $\zeta > 1$

Homog. $z_h(t) = e^{-\zeta w_n t} \left(A e^{w_n \sqrt{\zeta^2 - 1}\, t} + B e^{-w_n \sqrt{\zeta^2 - 1}\, t} \right)$

total $z(t) = e^{-\zeta w_n t} \left(A e^{w_n \sqrt{\zeta^2 - 1}\, t} + B e^{-w_n \sqrt{\zeta^2 - 1}\, t} \right) + A_b \cos w_b t + B_b \sin w_b t$

$$\Rightarrow x(t) = e^{-\zeta w_n t} \left(A e^{w_n \sqrt{\zeta^2 - 1}\, t} + B e^{-w_n \sqrt{\zeta^2 - 1}\, t} \right) + A_b \cos w_b t + B_b \sin w_b t + y(t)$$

Now, we get A and B from IC's.

$\boxed{\xi = 1}$

+5

$$x(t) = Ae^{-w_n t} + Bte^{-w_n t} + \alpha\cos wt + \beta \sin wt + y(t)$$

$$\dot{x}(t) = -w_n A e^{-w_n t} - B w_n t e^{-w_n t} - \alpha w \sin wt + \beta w \cos wt + \dot{y}(t)$$

$$IC = x(0) = x_0 = A + 0 + \alpha + 0 + y(0) \Rightarrow A = x_0 - \alpha - y(0)$$

$$\dot{x}(0) = v_0 = -w_n A - 0 - 0 + \beta w + \dot{y}(0) \Rightarrow B = v_0 + w_n A - \beta w - \dot{y}(0)$$

$\boxed{\xi > 1}$

$$x(t) = e^{-\xi w_n t}\left(A e^{w_n \sqrt{\xi^2-1}\, t} + B e^{-w_n t\sqrt{\xi^2-1}}\right) + \alpha \cos wt + \beta \sin wt + y(t)$$

$$\dot{x}(t) = -\xi w_n e^{-\xi w_n t}\left(A e^{w_n t\sqrt{\xi^2-1}} + B e^{-w_n \sqrt{\xi^2-1}\, t}\right) +$$

$$e^{-\xi w_n t}\left(A w_n \sqrt{\xi^2-1}\, e^{w_n t\sqrt{\xi^2-1}} - B w_n \sqrt{\xi^2-1}\, e^{-w_n t\sqrt{\xi^2-1}}\right) - \alpha w \sin wt +$$

$$\beta w \cos wt + \dot{y}(t)$$

$$IC = x(0) = x_0 = 1\left(A w_n \sqrt{\xi^2-1} - B w_n \sqrt{\xi^2-1}\right) + \beta w + y(0)$$

$$\dot{x}(0) = v_0 = -\xi w_n (A+B) + \left(A w_n \sqrt{\xi^2-1} - B w_n \sqrt{\xi^2-1}\right) - \beta w + \dot{y}(0)$$

$$\therefore A = \frac{1}{2\xi w_n}\left[\left(w_n\sqrt{\xi^2-1} + \xi w_n\right)(x_0 - \alpha) - (v_0 - \beta w)\right] - y(0)$$

$$\therefore B = \frac{1}{2\xi w_n}\left[\left(-w_n\sqrt{\xi^2-1} + \xi w_n\right)(x_0 - \alpha) + v_0 - \beta w\right] - \dot{y}(0)$$

$\boxed{\xi < 1}$

$$x(t) = e^{-\xi w_n t}\left(A \cos w_d t + B \sin w_d t\right) + \alpha \cos wt + \beta \sin wt + y(t)$$

$$\dot{x}(t) = -\xi w_n e^{-\xi w_n t}\left(A\cos w_d t + B \sin w_d t\right) + e^{-\xi w_n t}\left(-w_d A \sin w_d t + B w_d \cos w_d t\right) -$$

$$\alpha w \sin wt + \beta w \cos wt + \dot{y}(t)$$

$$IC \Rightarrow x(0) = x_0 = A + 0 + \alpha + y(0) \Rightarrow A = x_0 - \alpha - y(0)$$

$$\Rightarrow \dot{x}(0) = v_0 = -\xi w_n A + 0 + 0 + B w_d + \beta w + \dot{y}(0)$$

$$\therefore B = \frac{\xi w_n A}{w_d} - \frac{\beta w}{w_d} + \frac{v_0}{w_d} - \dot{y}(0)$$

Ali Samir Al-Dabbagh

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