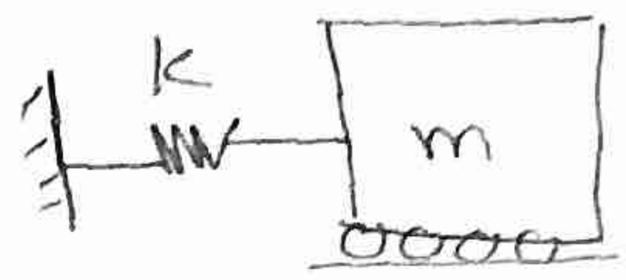
Chapter Two: Vibration of Discrete Systems: Brief Review

2.1 Vibration of Single-Degree-of-freedom System (SDOF)

2.1.1 (1) Undamped unforced Vibration (Free vibration)

Initial conditions
$x(0) = x_0$, ~~$v(0) = v_0$~~
$\dot{x}(0) = v_0$

Equation of motion (EOM)



$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \sqrt{\frac{k}{m}} = w_n \quad \text{natural Frequency (rad/sec)}$$

m: mass
k: Spring constant

$$\ddot{x} + w_n^2 x = 0$$

Solve

$$r^2 + w_n^2 = 0 \Rightarrow r = \pm \hat{i} w_n \qquad \hat{i} = \sqrt{-1}$$

$$\Rightarrow x(t) = A\cos w_n t + B\sin w_n t$$

Apply IC's

(1) $x(0) = x_0 = A \Rightarrow \boxed{A = x_0}$

(2) $\dot{x}(t) = -Aw_n \sin w_n t + Bw_n \cos w_n t$

$\dot{x}(0) = v_0 = Bw_n \Rightarrow B = \frac{v_0}{w_n}$

$$\Rightarrow x(t) = x_0 \cos w_n t + \frac{v_0}{w_n}\sin w_n t \quad \text{— Eq(1)}$$

or $x(t) = X\sin(w_n t + \phi) \longrightarrow$ How?

$$\sin(w_n t + \phi) = \sin w_n t \cos\phi + \cos w_n t \sin\phi$$

$$X\sin(w_n t + \phi) = \underbrace{X \sin w_n t \cos\phi}_{\frac{v_0}{w_n}} + \underbrace{X\cos w_n t \sin\phi}_{x_0} \quad \leftarrow \text{Compare to Eq(1)}$$

(a) $X\cos\phi = \frac{v_0}{w_n}$

(b) $X\sin\phi = x_0$

$$\Rightarrow \begin{aligned} X^2\cos^2\phi &= \left(\frac{v_0}{w_n}\right)^2 \\ + \quad X^2\sin^2\phi &= x_0^2 \end{aligned}$$

$$X^2(\underbrace{\sin^2\phi + \cos^2\phi}_{=1}) = x_0^2 + \left(\frac{v_0}{w_n}\right)^2$$

$$\Rightarrow X = \sqrt{x_0^2 + \left(\frac{v_0}{w_n}\right)^2}$$

$\frac{b}{a} \Rightarrow \tan\phi = \frac{x_0 \cdot w_n}{v_0}$

$$\phi = \tan^{-1}\left(\frac{x_0}{v_0} \cdot w_n\right)$$

Cont'd

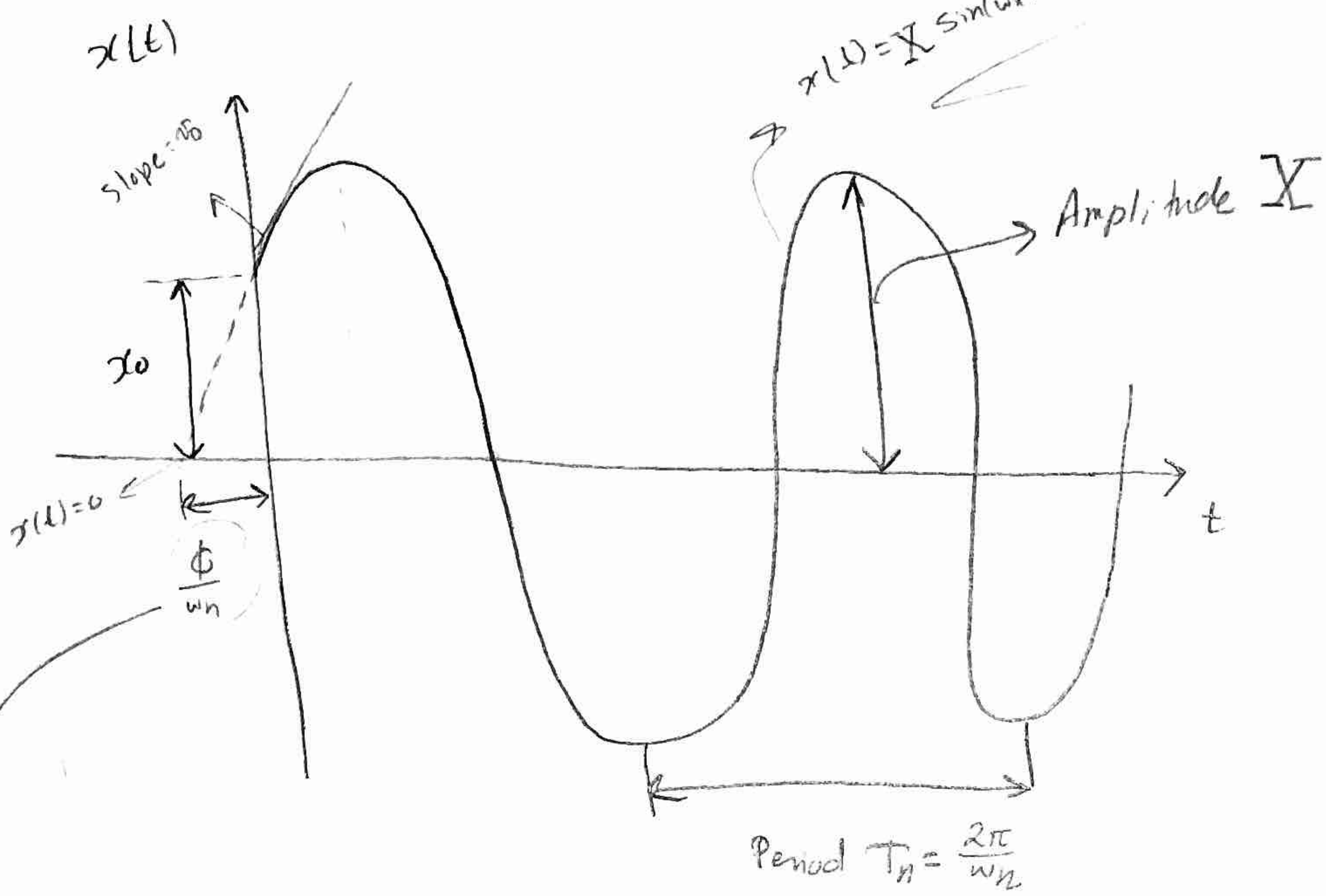
$$m\ddot{x} + kx = 0 \qquad \begin{array}{l} x(0) = x_0 \\ \dot{x}(0) = v_0 \end{array} \qquad , \quad w_n = \sqrt{\frac{k}{m}}$$

$$x(t) = x_0 \cos w_n t + \frac{v_0}{w_n} \sin w_n t$$

or

$$x(t) = X \sin(w_n t + \phi)$$

$$X = \sqrt{x_0^2 + \left(\frac{v_0}{w_n}\right)^2} \quad , \quad \phi = \tan^{-1}\left(\frac{x_0}{v_0} \cdot w_n\right)$$

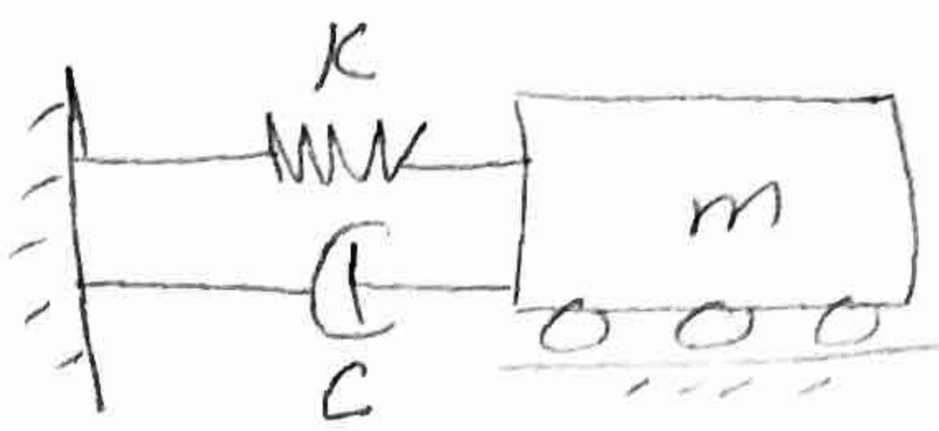


$$x(t) = 0 = X \sin(w_n t + \phi) \Rightarrow \sin(w_n t + \phi) = 0$$

$$w_n t + \phi = 0 \Rightarrow \boxed{t = \frac{-\phi}{w_n}}$$

m: mass
K: Spring constant
c: damping coefficient

(2) damped unforced (free) vibration



EOM

IC's. $x(0) = x_0$
$\dot{x}(0) = v_0$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$2\zeta\omega_n = \frac{c}{m} \Rightarrow \zeta = \frac{c}{2\omega_n m} = \frac{c}{2\sqrt{km}}$$ (unitless)

damping ratio

$$\omega_n^2 = \frac{k}{m} \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$ Nat. Frequency (rad/s)

Solution $\ddot{x} + 2\zeta\omega_n\dot{x} + kx = 0$

$ax^2 + bx + c = 0$

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$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0 \Rightarrow r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2} = \frac{-2\zeta\omega_n \pm 2\omega_n\sqrt{\zeta^2 - 1}}{2}$$

$$r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Damping cases

(1) $\zeta = 1 \Rightarrow r_{1,2} = -\omega_n$ [Critically damped]

(2) $\zeta > 1 \Rightarrow r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ [over damped]

(3) $\zeta < 1 \Rightarrow r_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2}$ [underdamped]

most available in application

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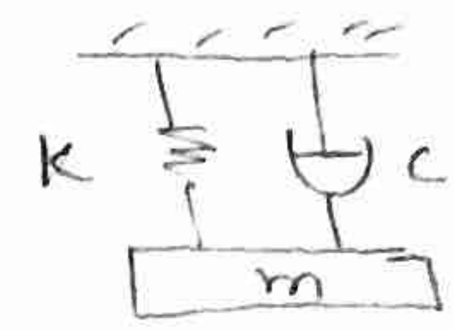
* Summary of Solution

* Free-damped SDOF



$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

"Teach this summary only"

① $\zeta = 1$

② $\zeta > 1$

③ $\zeta < 1$

① Critically damped $\zeta = 1$

$$x(t) = A e^{-\omega_n t} + B t e^{-\omega_n t}$$

$$A = x_0$$

$$B = v_0 + \omega_n x_0$$

② Overdamped $\zeta > 1$

$$x(t) = e^{-\zeta\omega_n t}\left(A e^{\omega_n\sqrt{\zeta^2-1}\,t} + B e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

$$A = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) + v_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$B = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right) - v_0}{2\omega_n\sqrt{\zeta^2-1}}$$

③ Underdamped $\zeta < 1$

$$x(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

or

$$x(t) = X\sin(\omega_d t + \phi)$$

$$X = \sqrt{(x_0)^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right)$$

$$A = x_0$$

$$B = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

damped natural Frequency

Cont'd

① **Critical damping** $\zeta = 1 \Rightarrow r_{1,2} = -w_n$ "repeated root" $r_{1,2} = r$

$$x(t) = A e^{rt} + Bt e^{rt}$$

IC's

① $x(0) = x_0 = A \Rightarrow \boxed{A = x_0}$

② $\dot{x}(t) = rAe^{rt} + rBte^{rt} + Be^{rt}$

$\dot{x}(t) = v_0 = rA + 0 + B \Rightarrow v_0 = -w_n x_0 + B$

$\Rightarrow \boxed{B = v_0 + x_0 w_n}$

$$\Rightarrow x(t) = x_0 e^{-w_n t} + (v_0 + x_0 w_n) t e^{-w_n t}$$

$$x(t) = \left[x_0 + (v_0 + x_0 w_n) t\right] e^{-w_n t}$$

Free, $\zeta = 1$

② **Over damped** $\zeta > 1$ $r_{1,2} = -\zeta w_n \pm w_n \sqrt{\zeta^2 - 1}$

$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = 0$ $x(0) = x_0$, $\dot{x}(0) = v_0$

$$x(t) = A e^{r_1 t} + B e^{r_2 t} = A e^{(-\zeta w_n + w_n \sqrt{\zeta^2 - 1})t} + B e^{(-\zeta w_n - w_n \sqrt{\zeta^2 - 1})t}$$

$$= A\left(e^{-\zeta w_n t} e^{w_n \sqrt{\zeta^2-1}\, t}\right) + B\left(e^{-\zeta w_n t} \cdot e^{-w_n \sqrt{\zeta^2-1}\, t}\right)$$

$$\Rightarrow x(t) = e^{-\zeta w_n t}\left(A e^{w_n \sqrt{\zeta^2-1}\, t} + B e^{-w_n \sqrt{\zeta^2-1}\, t}\right)$$

IC's

① $x(0) = x_0 \Rightarrow \boxed{A + B = x_0}$

cont'd

② $\dot{x}(0) = v_0$

$$\dot{x}(t) = e^{-\zeta w_n t}\left(A w_n\sqrt{\zeta^2-1}\, e^{w_n\sqrt{\zeta^2-1}\,t} - B w_n\sqrt{\zeta^2-1}\, e^{-w_n\sqrt{\zeta^2-1}\,t}\right)$$

$$- \zeta w_n e^{-\zeta w_n t}\left(A e^{w_n\sqrt{\zeta^2-1}\,t} + B e^{-w_n\sqrt{\zeta^2-1}\,t}\right)$$

$$\dot{x}(0) = v_0 = \left(A w_n\sqrt{\zeta^2-1} - B w_n\sqrt{\zeta^2-1}\right) - \zeta w_n (A+B)$$

Eq(2) $\Rightarrow$ $$v_0 = A\left(w_n\sqrt{\zeta^2-1} - \zeta w_n\right) + B\left(-w_n\sqrt{\zeta^2-1} - \zeta w_n\right)$$

Eq(1) $$x_0 = A + B$$

Matrix Form

$$\begin{bmatrix} 1 & 1 \\ w_n\sqrt{\zeta^2-1} - \zeta w_n & -w_n\sqrt{\zeta^2-1} - \zeta w_n \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

$$A = \frac{x_0 w_n \left(\zeta + \sqrt{\zeta^2-1}\right) + v_0}{2 w_n\sqrt{\zeta^2-1}}$$

$$B = \frac{-x_0 w_n \left(\zeta - \sqrt{\zeta^2-1}\right) - v_0}{2 w_n\sqrt{\zeta^2-1}}$$

$$x(t) = e^{-\zeta w_n t}\left(A e^{w_n\sqrt{\zeta^2-1}\,t} + B e^{-w_n\sqrt{\zeta^2-1}\,t}\right)$$

Cont'd

③ Under damped $\zeta < 1$ $\quad r_{1,2} = -\zeta w_n \pm \hat{i} w_n \sqrt{1-\zeta^2}$

$$r_{1,2} = -\zeta w_n \pm \hat{i} w_d, \quad w_d = w_n\sqrt{1-\zeta^2}$$

$$x(t) = A e^{r_1 t} + B e^{r_2 t} = \left(A e^{-\zeta w_n t} \cdot e^{+\hat{i} w_d t} + B e^{-\zeta w_n t} \cdot e^{-\hat{i} w_d t}\right)$$

$$= e^{-\zeta w_n t}\left(A e^{\hat{i} w_d t} + B e^{-\hat{i} w_d t}\right)$$

But $e^{i\theta} = \cos\theta + \hat{i}\sin\theta$ (Euler's Formula)

$$x(t) = e^{-\zeta w_n t}\left(A\left(\cos w_d t + i \sin w_d t\right) + B\left(\cos w_d t - \hat{i}\sin w_d t\right)\right)$$

$$= e^{-\zeta w_n t}\left[\underbrace{(A+B)}_{\alpha}\cos w_d t + \underbrace{i(A-B)}_{\beta}\sin w_d t\right]$$

$$x(t) = e^{-\zeta w_n t}\left[\alpha \cos w_d t + \beta \sin w_d t\right]$$

IC's

① $x(0) = x_0 \Rightarrow \boxed{\alpha = x_0}$

② $\dot{x}(0) = v_0$

$$\dot{x}(t) = e^{-\zeta w_n t}\left[w_d\, \alpha \sin w_d t + w_d \beta \cos w_d t\right]$$

$$- \zeta w_n e^{-\zeta w_n t}\left[\alpha \cos w_d t + \beta \sin w_d t\right]$$

$$\dot{x}(0) = v_0 = (w_d \beta) - \zeta w_n \overset{=x_0}{\alpha}$$

$$\Rightarrow \beta = \frac{v_0 + \zeta w_n x_0}{w_d}$$

$$x(t) = e^{-\zeta w_n t}\left(x_0 \cos w_d t + \frac{v_0 + \zeta w_n x_0}{w_d} \cdot \sin w_d t\right)$$

Cont'd

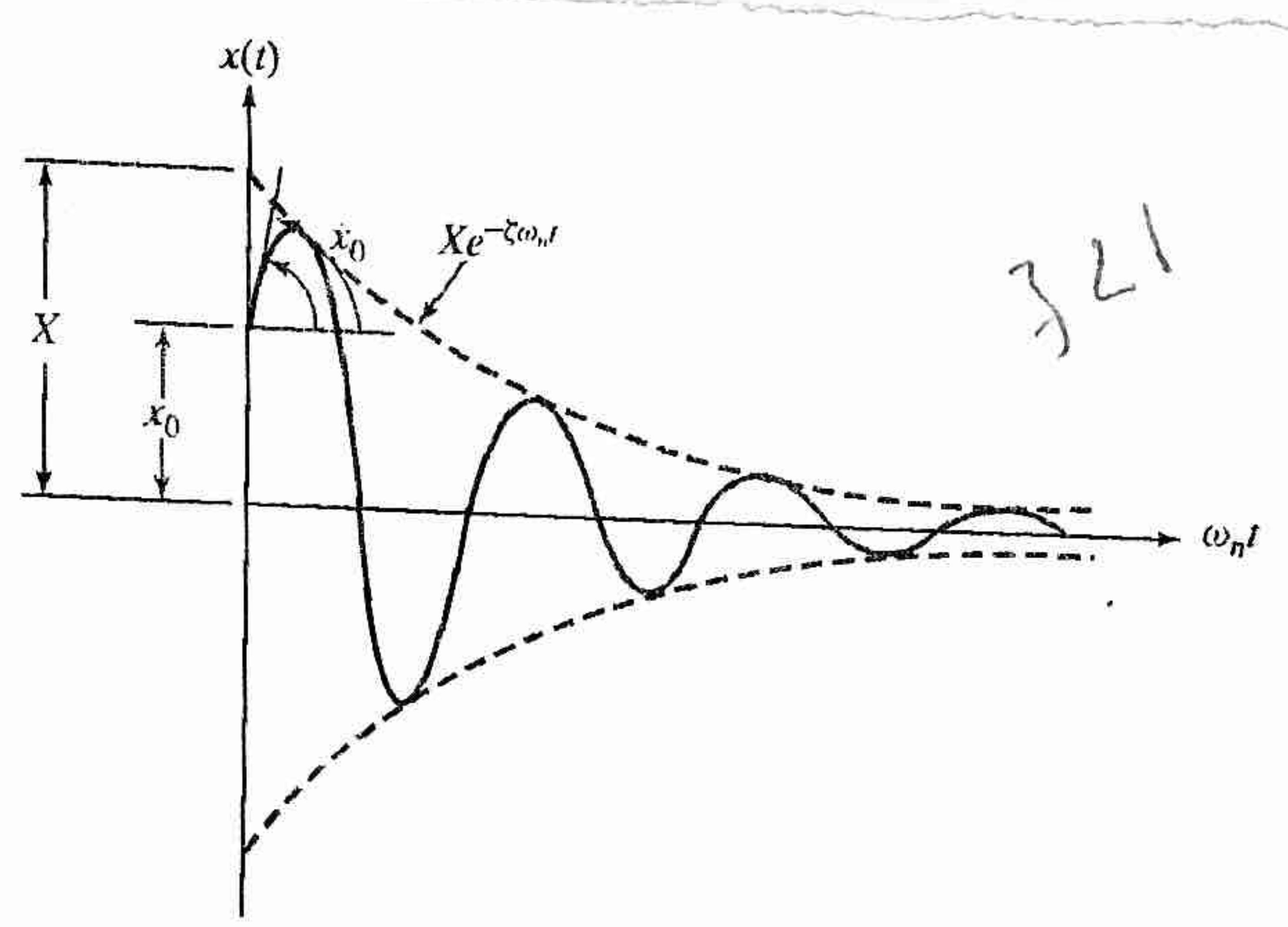
OR

$$x(t) = X \sin(\omega_d t + \phi) \quad , \quad \sin(\omega_d t + \phi) = \cos\omega_d t \sin\phi + \sin\omega_d t \cos\phi$$

$$= X \cos\omega_d t \sin\phi + X \sin\omega_d t \cos\phi$$

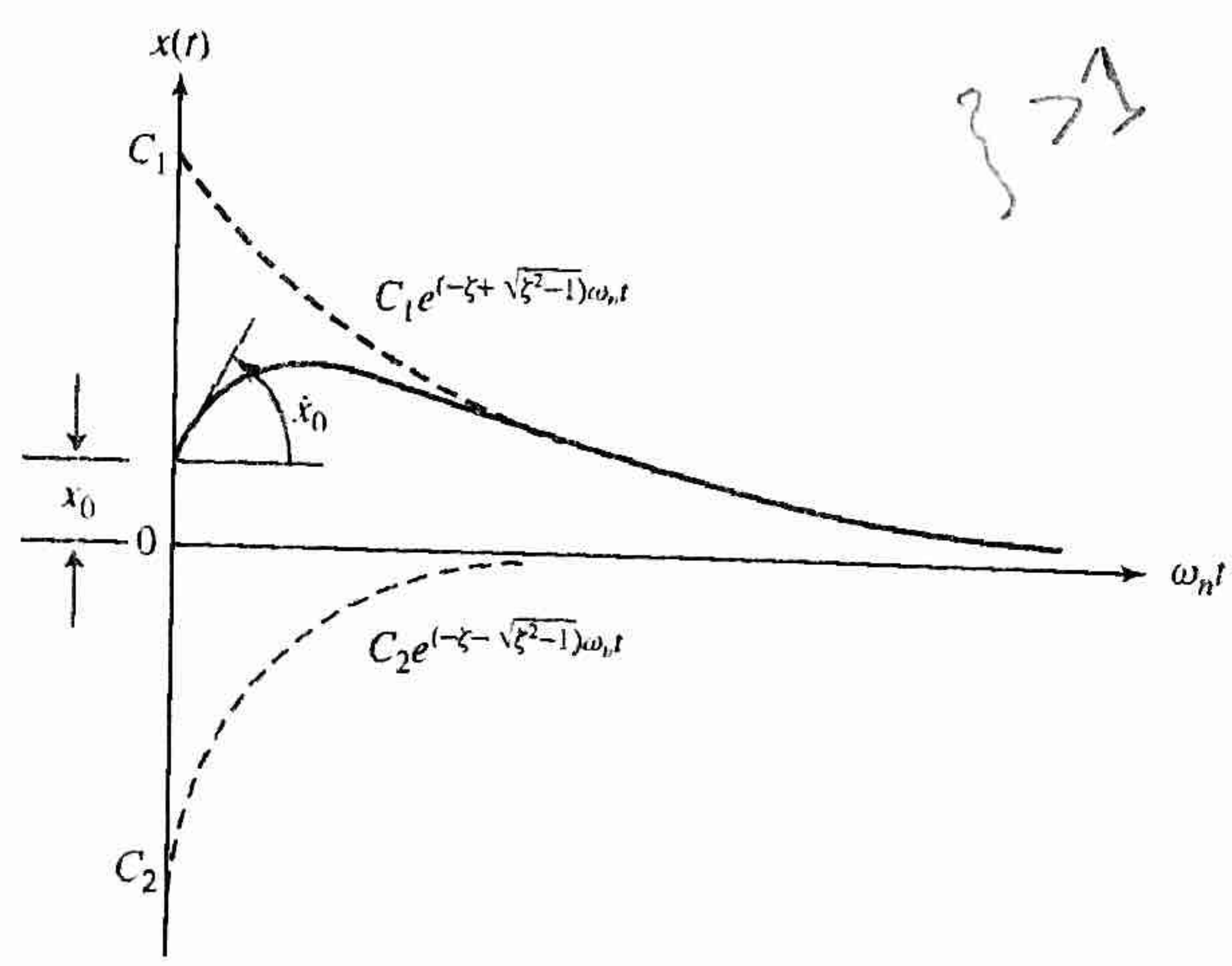
$$x_0 = X \sin\phi \quad , \quad \frac{v_0 + \zeta\omega_n x_0}{\omega_d} = X \cos\phi$$

$$X = \sqrt{(x_0)^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \quad , \quad \phi = \tan^{-1}\left(\frac{x_0 \omega_d}{v_0 + \zeta\omega_n x_0}\right)$$



(a)

$\zeta < 1$



(b)

$\zeta > 1$

$\zeta = 1$