Phys 741
Statistical Mechanics
Problem Set # 7

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1. Consider a 3D system of bosons, each of spin S, that is extremely relativistic with an energy spectrum $\varepsilon = cp$, where c is a constant.
   (a) Calculate the density of states of the system
   (b) Find the Bose-Einstein condensation temperature (the critical temperature T_c) at which the bosons start accumulating in the ground state, and show that the number of bosons in the excited states below T_c is given by $N_{ex} = N(\frac{T}{T_c})^3$
   (c) Find an expressions for the total energy E of the system for both $T > T_c$ and $T \lesssim T_c$
   (d) Find an expressions for the grand potential Ω of the system for both $T > T_c$ and $T \lesssim T_c$ and show that the pressure for $T > T_c$ is given by
   $$P = \frac{N}{V} k_B T \, \frac{g_4(z)}{g_3(z)}$$
   , and for $T \lesssim T_c$ is given by
   $$P = \frac{N}{V} k_B T_c \left(\frac{T}{T_c}\right)^4 \frac{\zeta(4)}{\zeta(3)}$$
   (e) Show that the entropy of the system for $T > T_c$ is given by
   $$\frac{S}{Nk_B} = 4\frac{g_4(z)}{g_3(z)} - \ln z$$
   , and for $T \lesssim T_c$ is given by
   $$\frac{S}{Nk_B} = 4\left(\frac{T}{T_c}\right)^3 \frac{\zeta(4)}{\zeta(3)}$$

2. The Bose Einstein integral is defined as
   $$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x - 1}$$
   (a) By expanding the integrand in powers of z, show that
   $$g_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n}$$
   and find $g_n(1)$
   (b) Show that for the case $z = 1$ ($\mu = 0$), the series in (a) converges for $n > 1$ and diverges for $n \leq 1$.
   (c) Show that for the case $z = 1$ ($\mu = 0$),
   $$\int_0^\infty \frac{x^n dx}{e^x - 1} = \Gamma(n+1)\ \zeta(n+1)$$
   This problem will help you answer the next problem

3. Consider a system of an ideal Bose gas confined to a 2D plane of area A
   (a) Calculate the density of states of the system
   (b) Show that the system does not exhibit Bose Einstein condensation unless $T \to 0$
4. Consider a system of an ideal Bose gas confined to a 1D line of length L
   (a) Calculate the density of states of the system
   (b) Show that the system does not exhibit Bose Einstein condensation unless $T \to 0$
5. Consider Debye model in 2D
   (a) Calculate the density of state of the system
   (b) Calculate the total energy and the heat capacity at constant volume of the system for both high T limit ($k_B T \gg \hbar\omega$) and low T limit ($k_B T \ll \hbar\omega$)

Good Luck